Existence and concentration of semiclassical states for nonlinear Schrödinger equations

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Abstract: In this paper, we study the following semilinear Schrödinger equation

$$-\epsilon^2\triangle u + u + V(x)u = f(u),\ u \in H^1(\mathbb{R}^N),$$

where $N \geq 2$ and $\epsilon > 0$ is a small parameter. The function V is bounded in $\mathbb{R}^N$, $\inf_{\mathbb{R}^N}(1+V(x)) > 0$ and it has a possibly degenerate isolated critical point. Under some conditions on f, we prove that as $\epsilon \to 0$, this equation has a solution which concentrates at the critical point of V.

Key words: semilinear Schrödinger equation, variational reduction method.
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1 Introduction and main result

In this paper, we are concerned with the following semilinear Schrödinger equation

$$-\epsilon^2\triangle u + u + V(x)u = f(u),\ u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $N \geq 2$ and $\epsilon > 0$ is a small parameter. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$(\mathbf{F_1})$. $f \in C^1(\mathbb{R})$ and there exist $q \in (2, 2^*)$, $2 < p_1 < p_2 < 2^*$ and a constant $C > 0$ such that

$$|f'(t)| \leq C(|t|^{p_1-2} + |t|^{p_2-2}),\ t \in \mathbb{R}$$

and for any $L > 0$,

$$\sup\{|f'(t) - f'(s)|/|t-s|^{q-2} \mid t, s \in [-L, L],\ t \neq s\} < \infty, \tag{1.2}$$

where $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$;

$(\mathbf{F_2})$. there exists $\mu > 2$ such that $f(t)t \geq \mu F(t) > 0$, $t \neq 0$, where $F(t) = \int_0^t f(s)ds$;

$(\mathbf{F_3})$. $f(t)/|t|$ is an increasing function on $\mathbb{R} \setminus \{0\}$;

Remark 1.1. *A typical function which satisfies* $(\mathbf{F_1}) - (\mathbf{F_3})$ *is*

$$f(t) = \sum_{i=1}^{m} a_i|t|^{\beta_i-2}t$$

with $2 < \beta_1 < \cdots < \beta_m < 2^*$ *and* $a_i > 0$, $1 \leq i \leq m$.

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The potential function V satisfies the following conditions:

$(\mathbf{V_0})$. $\inf_{x\in\mathbb{R}^N}(1+V(x))>0$ and $\max_{x\in\mathbb{R}^N}|V(x)|<\infty$;

$(\mathbf{V_1})$. $V\in C^2(\mathbb{R}^N)$ has an isolated critical point x_0 such that

$$V(x)=Q_{n^*}(x-x_0)+o(|x-x_0|^{n^*})$$

in some neighborhood of x_0, where $n^*\geq 2$ is an even integer and Q_{n^*} is an n^*- homogeneous polynomial in $\mathbb{R}^N$ which satisfies that $\triangle Q_{n^*}\geq 0$ in $\mathbb{R}^N$ or $\triangle Q_{n^*}\leq 0$ in $\mathbb{R}^N$ and $\triangle Q_{n^*}\not\equiv 0$ in $\mathbb{R}^N$.

Remark 1.2. *Without loss of generality, in what follows, we always assume that $x_0=0$. Typical examples for Q_{n^*} are $\pm|x|^{n^*}$ $(n^*\geq 2)$.*

Our main result of this paper is the following theorem

Theorem 1.3. *Suppose that f satisfies $(\mathbf{F_1})-(\mathbf{F_4})$ and V satisfies $(\mathbf{V_0})$ and $(\mathbf{V_1})$. Then there exist $\epsilon_0>0$ and a set $\mathcal{K}$ whose elements are radially symmetric solutions of equation*

$$-\triangle u+u=f(u),\ u\in H^1(\mathbb{R}^N) \tag{1.3}$$

such that if $0<\epsilon<\epsilon_0$, then equation (1.1) has a solution u_ϵ satisfying that

$$\lim_{\epsilon\to 0} dist_{_Y}(v_\epsilon,\mathcal{K})=0,$$

where $v_\epsilon(x)=u_\epsilon(\epsilon x)$, $x\in\mathbb{R}^N$ and $Y=H^1(\mathbb{R}^N)$.

The analysis of the semilinear Schrödinger equation (1.1) has recently attracted a lot of attention due to its many applications in mathematical physics.

If v is a solution of equation (1.1), then $v(\epsilon x)$ is a solution of the following equation

$$-\triangle u+u+V(\epsilon x)u=f(u),\ u\in H^1(\mathbb{R}^N). \tag{1.4}$$

Equation (1.4) is a perturbation of the limit equation (1.3). If equation (1.3) has a solution $w\in C^2(\mathbb{R}^N)$ satisfying the non-degeneracy condition:

$$\ker L_0=\text{span}\left\{\frac{\partial\omega}{\partial x_i}\mid 1\leq i\leq N\right\},$$

where $L_0v=-\triangle v+v-f'(\omega)v$, then in the celebrated paper [1] (see also [2]), Ambrosetti, Badiale and Cingolani developed a kind of variational reduction method and showed that if the potential function V has a strictly local minimizer or maximizer x_0, then equation (1.4) admits a solution u_ϵ which converges to $\omega(\cdot-x_0)$ in $H^1(\mathbb{R}^N)$ as $\epsilon\to 0$. In their argument, the non-degeneracy property of ω plays essential role. Using the non-degeneracy condition and the reduction method, it was shown by Kang and Wei [20] that, at a strict local maximum point x_0 of V and for any positive integer k, (1.1) has a positive solution with k interacting bumps concentrating near x_0, while at a non-degenerate local minimum point of $V(x)$ such solutions do not exist. Moreover, under the assumption of the non-degeneracy condition, multiplicity of solutions with one bump has also been considered by Grossi [16].

However, for a general nonlinearity f, it is very difficult to verify the non-degeneracy condition for a solution of (1.3). An effective method to attack problem (1.1) without using the non-degeneracy condition is variational method. In [21], Rabinowitz used a global variational method to show the existence of least energy solutions for (1.1) when $\epsilon>0$ is small, and the condition imposed on V is a global one, namely

$$0<\inf_{x\in\mathbb{R}^N}(1+V(x))<\liminf_{|x|\to\infty}(1+V(x)).$$

In [12], [13], [14], [15] and [17], Del Pino, Felmer and Gui used different variational methods to obtain nontrivial solution of (1.1) for small $\epsilon > 0$ under local conditions which can be roughly described as follows: V is local Hölder continuous on $\mathbb{R}^N$,

$$\inf_{x\in\mathbb{R}^N}(1+V(x)) > 0 \tag{1.5}$$

and there exists k disjoint bounded regions $\Omega_1, \cdots, \Omega_k$ in $\mathbb{R}^N$ such that

$$\inf_{x\in\partial\Omega_i} V(x) > \inf_{x\in\Omega_i} V(x). \tag{1.6}$$

Their methods involve the deformation of nonlinearity f and some prior estimates. Recently, Byeon, Jeanjean and Tanaka [5] [6] developed the variational methods and made great advance in problem (1.1). Byeon and Jeanjean showed in [5] that if $N \geq 3$, V satisfies (1.5) and (1.6) with $k = 1$ and f satisfies

($\mathbf{f_1}$). $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{t\to 0+} f(t)/t = 0$;

($\mathbf{f_2}$). there exists some $p \in (1, 2^* - 1)$ such that $\lim_{t\to\infty} f(t)/t^p < \infty$;

($\mathbf{f_3}$). there exists $T > 0$ such that $\frac{1}{2}mT^2 < F(T)$, where $F(t) = \int_0^t f(s)ds$ and $m = \inf_{x\in\Omega_1} V(x)$,

then (1.1) exists positive solution v_ϵ concentrating in the minimizers of V in Ω_1 as $\epsilon \to 0$. And in [6], Byeon, Jeanjean and Tanaka considered the case $N = 1, 2$ and obtained similar results. Their conditions on the nonlinearity f are almost optimal. Moreover, when V satisfies (1.5) and (1.6) with $k > 1$ and f satisfies $(\mathbf{f_1})-(\mathbf{f_3})$, in [10], Cingolani, Jeanjean and Secchi constructed multi-bump solutions for magnetic nonlinear Schödinger equations which contain equation (1.1) as a special case.

Comparing to the variational methods mentioned above, the Lyapunov reduction method of Ambrosetti and Badiale, although it need the non-degeneracy condition, has its advantages that their method can be used to deal with elliptic equations involving critical Sobolev exponent (see, for example, [3]) and other problems involving concentration compactness (see, for example, [18]).

In this paper, we indent to attack the problem (1.1) though a Lyapunov reduction method, but avoiding the non-degeneracy condition for the solutions of limit equation (1.3). In this paper, we develop a new reduction method for an isolated critical set $\mathcal{K}$ of the functional corresponding to (1.3). This method can be regarded as a generalization of Ambrosetti and Badiale's method. The non-degeneracy conditions for the solutions in this critical set are no longer necessary and it does not involve the deformation of nonlinearity. By combination of the new reduction method and Conley index theory which was developed by Chang and Ghoussoub in [9](see also [8]), we obtain a solution of (1.4) in a neighborhood of $\mathcal{K}$ for sufficiently small $\epsilon > 0$. Our method is new and it can be used to other problems which involve concentration compactness. In contrast with the results of Byeon, Jeanjean and Tanaka, although the assumptions we imposed on the nonlinearity f are much stronger, the assumptions we made on V seem weaker in a sense, because by the assumption $(\mathbf{V_1})$, x_0 can be a local maximum point of V.

This paper is organized as follows: In section 2, we obtain a critical set of the functional corresponding to (1.3) with nontrivial Topology. In section 3 and section 4, a reduction for the function corresponding to (1.4) is developed. In section 5, we give the proof of Theorem 1.3. Section 6 and 7 are appendixes.

Notations. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of real number, integer and positive integer respectively. Let E be a metric space. $B_E(a, \rho)$ denotes the open ball in E centered at a and having radius ρ. The closure of a set $A \subset E$ is denoted by $\overline{A}$ or $cl_E(A)$. $\mathrm{dist}_E(a, A)$ denotes the distance from the point a to the set $A \subset E$. By $\to$ we denote the strong and by $\rightharpoonup$ the weak convergence. By $\ker A$ denotes the null space of the operator A. If g is a C^2 functional defined on a Hilbert space H, ∇g (or Dg) and $\nabla^2 g$ (or $D^2 g$) denote the gradient of g and the second derivative of g respectively. And for $a, b \in \mathbb{R}$, we denote $g^a := \{u \in H \mid g(u) \leq a\}$ and $g_b := \{u \in H \mid g(u) \geq b\}$ the sub- and super-level sets of the functional g, moreover, $g_b^a := \{u \in H \mid b \leq g(u) \leq a\}$. $\delta_{i,j}$ denotes the Kronecker notation, i.e., $\delta_{i,j} = 1$ if $i = j$ and 0 if $i \neq j$. For a Banach space E, denote $\mathcal{L}(E)$ the Banach space consisting of all bounded linear operator from E to E. If H is a Hilbert space and W is a closed subspace of H, we denote the orthogonal complement space of W in H by $W^\perp$. For a subset $A \subset H$, span$\{A\}$ denotes the subspace of H generated by A. For a topology pair (A, B) in metric space, $\check{H}^*(A, B)$ denotes the Čech-Alexander-Spanier cohomology with coefficient group $\mathbb{Z}_2$ (see [23]).

2 Critical sets of limit functional with nontrivial Topology

Throughout this paper, we denote the Sobolev space $H^1(\mathbb{R}^N)$ and the radially symmetric function space

$$H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \mid u \text{ is radially symmetric}\}$$

by Y and X respectively. The inner product of Y is

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)dx,$$

and we use $||\cdot||$ to denote the norm of Y corresponding to this inner product. Define

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx - \int_{\mathbb{R}^N} F(u)dx,\ u \in X.$$

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx - \int_{\mathbb{R}^N} F(u)dx,\ u \in Y,$$

$$E_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2 + V(\epsilon x)|u|^2)dx - \int_{\mathbb{R}^N} F(u)dx,\ u \in Y.$$

For $h \in H^{-1}(\mathbb{R}^N)$, let $(-\triangle + 1)^{-1}h$ and $(-\triangle + 1 + V(\epsilon x))^{-1}h$ be the solutions of

$$-\triangle u + u = h,\ u \in H^1(\mathbb{R}^N) \tag{2.1}$$

and

$$-\triangle u + u + V(\epsilon x)u = h,\ u \in H^1(\mathbb{R}^N) \tag{2.2}$$

respectively.

Under conditions $(\mathbf{F_1}) - (\mathbf{F_3})$, I satisfies Palais-Smale condition (see, for example, [24]) and has a mountain pass geometry, that is,

(i) $I(0) = 0$,

(ii) there exist $\rho_0 > 0$ and $\delta_0 > 0$ such that $I(u) \geq \delta_0$ for all $||u|| = \rho_0$,

(iii) there exists $u_0 \in X$ such that $||u_0|| > \rho_0$ and $I(u_0) < 0$.

Thus the following minimax value is well defined and is larger than δ_0,

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \tag{2.3}$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0,\ I(\gamma(1)) < 0\}. \tag{2.4}$$

Lemma 2.1. *For any $\sigma \in (0, \delta_0)$, if $a \in (c - \sigma, c)$ and $b \in (c, c + \sigma)$ are regular values of I, then $\check{H}^1(I^b, I^a) \neq 0$.*

Proof. Since $b > c$, by the definition of minimax value c, there exists $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} I(\gamma(t)) < b. \tag{2.5}$$

Let $u_0 = \gamma(1)$. We infer that 0 and u_0 lie in different connected component of I^a. It follows that the homomorphism

$$\iota^* : \check{H}^0(I^a) \to \check{H}^0(\{0, u_0\}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

which is induced by the inclusion mapping $\iota : \{0, u_0\} \hookrightarrow I^a$ is a surjection. Consider the following homomorphism which is induced by the inclusion mapping $j : \{0, u_0\} \hookrightarrow I^b$,

$$j^* : \check{H}^0(I^b) \to \check{H}^0(\{0, u_0\}).$$

By (2.5), 0 and u_0 lie in the same connected component of I^b. It follows that j^* is not a surjection.

Consider the following communicative diagram

$$\begin{tikzcd}
{} \arrow[r] & \check{H}^0(I^b) \arrow[r, "i^*"] \arrow[dr, "j^*"'] & \check{H}^0(I^a) \arrow[r, "\alpha^*"] \arrow[d, "\iota^*"] & \check{H}^1(I^b, I^a) \arrow[r] & {} \\
 & & \check{H}^0(\{0, u_0\}) & &
\end{tikzcd}$$

Since j^* is not a surjection and ι^* is a surjection, by this communicative diagram, we deduce that Image$(i^*) \neq \check{H}^0(I^a)$. Moreover, by the property of exact sequence, we have Image$(i^*) = \ker \alpha^*$. Thus $\ker \alpha^* \neq \check{H}^1(I^a)$. It follows that $\alpha^* \neq 0$. Therefore, $\check{H}^1(I^b, I^a) \neq 0$. □

From Chapter 4 of [24], we have the following lemma

Lemma 2.2. *If $\nabla I(u) = 0$ and $I(u) < 2c$, then u does not change sign in $\mathbb{R}^N$.*

Let $\mathcal{F}$ be a C^1 functional defined on a Hilbert space M with critical set $K_{\mathcal{F}}$. And let V be a pesudo-gradient vector field with respect to $D\mathcal{F}$ on M. A pesudo-gradient flow associated with V is the unique solution of the following ordinary differential equation in M :

$$\dot{\eta} = -V(\eta(x, t)), \ \eta(x, 0) = x.$$

A subset W of M is said to have the mean value property (for short (MVP)) if for any $x \in M$ and any $t_0 < t_1$ we have $\eta(x, [t_0, t_1]) \subset W$ whenever $\eta(x, t_i) \in W, \ i = 1, 2$.

Definition 2.3. *(Definition I.10 of [9]) Let $\mathcal{F}$ be a C^1 functional on a Hilbert space M. A subset S of the critical set K of $\mathcal{F}$ is said to be a dynamically isolated critical set if there exist a closed neighborhood $\mathcal{O}$ of S and regular values $a < b$ of $\mathcal{F}$ such that*

$$\mathcal{O} \subset \mathcal{F}^{-1}[a, b] \tag{2.6}$$

and

$$cl(\widetilde{\mathcal{O}}) \cap K \cap \mathcal{F}^{-1}[a, b] = S, \tag{2.7}$$

where $\widetilde{\mathcal{O}} = \bigcup_{t \in \mathbb{R}} \eta(\mathcal{O}, t)$. $(\mathcal{O}, a, b)$ is called an isolating triplet for S.

Definition 2.4. *(Definition III.1 of [9]) Let $\mathcal{F}$ be a C^1 functional on a Hilbet space M and let S be a subset of the critical set $K_{\mathcal{F}}$ for $\mathcal{F}$. A pair (W, W_-) of subset is said to be a GM pair for S associated with a pesudo-gradient vector field V, if the following conditions hold:*

(1). W is a closed (MVP) neighborhood of S satisfying $W \cap K = S$ and $W \cap \mathcal{F}_\alpha = \emptyset$ for some α.

(2). W_- is an exit set for W, i.e., for each $x_0 \in W$ and $t_1 > 0$ such that $\eta(x_0, t_1) \notin W$, there exists $t_0 \in [0, t_1)$ such that $\eta(x_0, [0, t_0]) \subset W$ and $\eta(x_0, t_0) \in W_-$.

(3). W_- is closed and is a union of a finite number of sub-manifolds that transversal to the flow η.

For $\alpha, \beta \in \mathbb{R}$, define

$$\mathcal{K}_\alpha^\beta := \{u \in X \mid \nabla I(u) = 0, \ \alpha \leq I(u) \leq \beta\}.$$

Let a and b are the regular values which come from Lemma 2.1. Then by Definition 2.4, $\mathcal{K}_a^b$ is a dynamically isolated critical set of I. By Lemma 2.1 and Theorem III.3 of [9], we have the following lemma

Lemma 2.5. *Let $\sigma > 0$ be sufficiently small and $a \in (c - \sigma, c)$, $b \in (c, c + \sigma)$ be regular values of I. If (W, W_-) is a GM pair of $\mathcal{K}_a^b$ associated with some pseudo-gradient vector field of I, then*

$$\check{H}^1(W, W_-) \neq 0.$$

Remark 2.6. *In this remark, we shall show that the set of regular values of I is dense in $\mathbb{R}$. Therefore, for any $\sigma > 0$, there always exist regular values of I in $(c-\sigma, c)$ and $(c, c+\sigma)$. In fact, we shall show that $I(C)$ is of first category, where C is the set of critical points of I. It suffices to prove that for any $u \in C$, there exists $\delta_u > 0$ such that $\overline{I(C \cap B_X(u, \delta_u))}$ does not contain interior points.*

Let $u \in C$. Since u is radially symmetric, the dimension of the kernel space of the following operator is at most one

$$\nabla^2 I(u) : X \to X,\ h \in X \mapsto h - (-\triangle + 1)^{-1} f'(u)h.$$

If $\dim \nabla^2 I(u) = 0$, then by Morse Lemma (see, e.g., Lemma 4.1 of [7]), there exists $\delta_u > 0$ such that u is the unique critical point of I in $B_X(u, \delta_u)$. Thus, in this case, $I(C \cap B_X(u, \delta_u)) = \{I(u)\}$.

If $\dim \nabla^2 I(u) = 1$, let $N = ker \nabla^2 I(u)$ and note that I is a C^2 functional, then by Lemma 1 of [19] (see also Theorem 5.1 of [7]), there exist an origin preserving C^1 diffeomorphism Φ of some $B_X(0, \delta_u)$ into X and an an origin preserving C^1 map h defined in $N \cap B_X(0, \delta_u)$ into X such that

$$I \circ \Phi(z, y) = I(u) + ||Pz||^2 - ||(id - P)z||^2 + I(h(y) + y)$$

where $P : N^\perp \to N^\perp$ is an orthogonal projection and $N^\perp$ is the orthogonal complement of N in X. Let $U = \{y \in N \cap B_X(0, \delta_u) \mid h(y) + y\}$. Then U is a C^1 one-dimensional manifold. Let us restrict I to U. Then $I : U \to \mathbb{R}$ is C^1. Moreover, $C \cap B_X(0, \delta_u) = C \cap U$, so $I(C \cap B_X(0, \delta_u)) = I(C \cap U)$. Therefore, by classical Sard theorem, $\overline{I(C \cap B_X(0, \delta_u))}$ does not contain interior points.

For $r > 0$, $A \subset X$, let

$$N_r(A) := \{v \in X \mid \text{dist}_X(v, A) < r\}. \tag{2.8}$$

Lemma 2.7. *Let c be the mountain pass value coming from Lemma 2.1. For any $r > 0$, there exists $\sigma_r > 0$ such that if $a \in (c - \sigma_r, c)$ and $b \in (c, c + \sigma_r)$ are regular values of I, then there exists a GM pair (W, W_-) of the critical set $\mathcal{K}_a^b$ of the functional I associated with the negative gradient vector field of I such that $W \subset N_r(\mathcal{K}_a^b)$.*

Proof. By $(\mathbf{F_1}) - (\mathbf{F_3})$, we know that I satisfies the Palais-Smale condition (see [24]). Therefore, for any $r > 0$, there exists $\kappa_r > 0$ such that if $a \in (c-1, c)$ and $b \in (c, c+1)$, then

$$||\nabla I(v)|| \geq \kappa_r,\ \forall v \in I^{-1}[a, b] \setminus N_{r/3}(\mathcal{K}_a^b). \tag{2.9}$$

Let

$$0 < \sigma_r < \min\{r\kappa_r/6, 1\} \tag{2.10}$$

and $a \in (c - \sigma_r, c)$ and $b \in (c, c + \sigma_r)$ be regular values of I. For

$$u \in I^{-1}[a, b] \cap N_{r/3}(\mathcal{K}_a^b), \tag{2.11}$$

consider the negative gradient flow:

$$\dot\eta(t) = -\nabla I(\eta(t)),\ \eta(0) = u. \tag{2.12}$$

Let

$$T_u^+ = \sup\{t \geq 0 \mid \text{for every } s \in [0, t],\ I(\eta(s)) \geq a\}$$

and

$$T_u^- = \inf\{t \leq 0 \mid \text{for every } s \in [t, 0],\ I(\eta(s)) \leq b\}.$$

Let

$$U = \bigcup_{t \in [T_u^-, T_u^+]} \{\eta(t, u) \mid u \in I^{-1}[a, b] \cap N_{r/3}(\mathcal{K}_a^b)\}.$$

Then

$$[\mathcal{K}_a^b] \subset U,$$

where

$$[\mathcal{K}_a^b] = \{v \in X \mid \omega(v) \cup \omega^*(v) \in \mathcal{K}_a^b\},$$

$\omega(v) = \cap_{t>0}\overline{\eta(v,[t,+\infty))}$ is the $\omega-$limit set of v and $\omega^*(v) = \cap_{t>0}\overline{\eta(v,(-\infty,-t])}$ is the ω^*-limit set of v.

By [9, Proposition III.2], we deduce that there exists a GM pair (W, W_-) of $\mathcal{K}_a^b$ such that $W \subset U$. Thus, to prove this Lemma, it suffices to prove that if $\sigma_r > 0$ is small enough, then for u which satisfies (2.11),

$$\sup_{t \in (T_u^-, T_u^+)} ||\eta(t) - u|| \le \frac{2}{3} r. \tag{2.13}$$

Since their arguments are similar, we only give the proof for

$$\sup_{t \in [0, T_u^+)} ||\eta(t) - u|| \le \frac{2}{3} r. \tag{2.14}$$

If (2.14) were not true, then there exist $0 \le t_1 < t_2 < T_u^+$ such that

$$r/3 \le ||\eta(t) - u|| \le 2r/3, \ \forall t \in [t_1, t_2]$$

$$||\eta(t_1) - u|| = r/3, \ ||\eta(t_2) - u|| = 2r/3. \tag{2.15}$$

According to (2.9), we have

$$\begin{aligned} & b - a \ge I(\eta(t_1)) - I(\eta(t_2)) \\ & = \int_{t_2}^{t_1} \langle \nabla I(\eta(t)), \dot{\eta}(t) \rangle dt = \int_{t_1}^{t_2} ||\nabla I(\eta(t))||^2 dt \ge \kappa_r^2 (t_2 - t_1). \end{aligned}$$

It follows that

$$t_2 - t_1 \le (b-a)/\kappa_r^2. \tag{2.16}$$

Combining (2.15) and (2.16) leads to

$$\begin{aligned} \frac{r}{3} &\le ||\eta(t_2) - \eta(t_1)|| \le \int_{t_1}^{t_2} ||\dot{\eta}(t)|| dt \\ &\le (t_2 - t_1)^{1/2} (\int_{t_1}^{t_2} ||\dot{\eta}(t)||^2)^{1/2} = (t_2 - t_1)^{1/2} (\int_{t_1}^{t_2} ||\nabla I(\eta(t))||^2)^{1/2} \\ &\le (t_2 - t_1)^{1/2} (b-a)^{1/2} \le (b-a)/\kappa_r < 2\sigma_r/\kappa_r. \end{aligned}$$

It contradicts (2.10). Thus, (2.14) holds. □

3 A variational reduction for the limiting functional I

Let $\sigma > 0$ be sufficiently small and $a \in (c - \sigma, c)$, $b \in (c, c + \sigma)$ be regular values of I, where c is defined by (2.3). In what follows, for the sake of simplicity, we denote the critical set $\mathcal{K}_a^b$ by $\mathcal{K}$.

By [4], if $u \in Y$ is a weak solution of

$$-\triangle u + u = f(u), \tag{3.1}$$

then u and $\frac{\partial u}{\partial x_i}$, $1 \le i \le N$ satisfy exponential decay at infinity. As a consequence, $\mathcal{K}$ is a compact subset of $W^{2,2}(\mathbb{R}^N)$. If $u \in Y$ is a solution of equation (3.1), then $\frac{\partial u}{\partial x_i}$, $i = 1, \cdots, N$ are the eigenfunctions for the eigenvalue problem

$$-\triangle h + h = f'(u) h. \tag{3.2}$$

Remark 3.1. *By [22, Theorem C. 3.4]), any eigenfunction of the eigenvalue problem (*3.2*) satisfies exponential decay at infinity.*

The argument in [11, Page 970-971] implies the following Lemma.

Lemma 3.2. *Suppose that $u \in X$ is a solution of equation (*3.1*) and it does not change sign in $\mathbb{R}^N$. If $v \in Y$ is a solution of (*3.2*) and satisfies*

$$\left\langle v, \frac{\partial u}{\partial x_i} \right\rangle = 0,\ i = 1, \cdots, N,$$

then $v \in X$.

Remark 3.3. *By Lemma* 2.2*, we infer that if $u \in \mathcal{K}$, then u does not change sign in $\mathbb{R}^N$.*

As it has been mentioned above, $\mathcal{K}$ is a compact subset in $W^{2,2}(\mathbb{R}^N)$. Thus for any $u \in \mathcal{K}$ and any $\varsigma > 0$, there exists $\tau_u > 0$ such that

$$\sum_{j=1}^{N} \left|\left| \frac{\partial v}{\partial x_j} - \frac{\partial u}{\partial x_j} \right|\right| < \varsigma,\ \forall v \in \mathcal{K} \cap B_X(u, 2\tau_u). \tag{3.3}$$

Therefore, we can choose a finite open sub-covering of $\mathcal{K}$

$$\mathcal{A} = \{B_X(u_i, \tau_{u_i}) \mid i = 1, \cdots, s\} \tag{3.4}$$

from the open covering $\{B_X(u, \tau_u) \mid u \in \mathcal{K}\}$. Let $\zeta \in C^\infty([0, +\infty))$ be such that $0 \leq \zeta(t) \leq 1$ for all t, $\zeta(t) = 1$ for $t \in [0, 1/2]$ and $\zeta(t) = 0$ for $t \in [1, \infty)$. Let

$$\xi_i(u) = \frac{\zeta(||u - u_i||/\tau_{u_i})}{\sum_{i=1}^{s} \zeta(||u - u_i||/\tau_{u_i})},\ 1 \leq i \leq s.$$

Then $\{\xi_i \mid 1 \leq i \leq s\}$ is a C^∞ partition of unity corresponding to the covering $\mathcal{A}$.

For $u \in \mathcal{K}$, let

$$Y_u := \{h \in X \mid \nabla^2 I(u)h = 0\},\ Z_u := \text{span}\{\frac{\partial u}{\partial x_i} \mid 1 \leq i \leq N\}.$$

Let

$$\mathcal{Y} = \text{span}\{\cup_{i=1}^{s} Y_{u_i}\}. \tag{3.5}$$

Let

$$q = \dim \mathcal{Y}. \tag{3.6}$$

Let $\{e_1, e_2, \cdots, e_q\}$ be an orthogonal normal base of $\mathcal{Y}$. As mentioned in Remark 3.1, for every $1 \leq n \leq q$, $e_n \in W_r^{2,2}(\mathbb{R}^N)$ and e_n satisfies exponential decay at infinity.

Let $\{e_1', e_2' \cdots\}$ be an orthogonal normal base of $\mathcal{Y}^\perp$, where $\mathcal{Y}^\perp$ is the orthogonal complement space of $\mathcal{Y}$ in X. From the appendix A of this paper, for every $k \in \mathbb{N}$, there exists

$$E_k := \{\tilde{e}_{j,k} \mid 1 \leq j \leq k\}, \tag{3.7}$$

such that

(i) For every k, $E_k \subset X \cap W_r^{2,2}(\mathbb{R}^N)$ and $E_k \perp \mathcal{Y}$;

(ii) Every $\tilde{e}_{j,k}$ satisfies exponential decay at infinity, $\langle \tilde{e}_{j,k}, \tilde{e}_{j',k} \rangle = \delta_{j,j'}$ and

$$\sup_{1 \leq j \leq k} ||\tilde{e}_{j,k} - e_j'|| \leq 1/2^k.$$

For every k, denote

$$X_k := \text{span}\{E_k\} \oplus \mathcal{Y}.$$

Let $P_k : X \to X_k$ and $P_k^\perp : X \to X_k^\perp$ be the orthogonal projections, where $X_k^\perp$ is the orthogonal complement space of X_k in X. By the definition of X_k and the properties (**i**) and (**ii**) mentioned above, we have the following Lemma which is easy to prove.

Lemma 3.4. *For every $h \in X$, $\lim_{k\to\infty} ||h - P_k h|| = \lim_{k\to\infty} ||P_k^\perp h|| = 0$.*

Lemma 3.5. *For any $r > 0$, there exists $l_r \in \mathbb{N}$ such that if $k \geq l_r$, then for every $v \in N_r(\mathcal{K})$, $P_k^\perp \nabla^2 I(v)|_{X_k^\perp}$ is invertible and*

$$||(P_k^\perp \nabla^2 I(v)|_{X_k^\perp})^{-1}||_{\mathcal{L}(X_k^\perp)} \leq 2.$$

Proof. For $w \in X_k^\perp$,

$$P_k^\perp \nabla^2 I(v) w = w - P_k^\perp (-\triangle + 1)^{-1} f'(v) w.$$

Denote the operator $w \mapsto P_k^\perp (-\triangle + 1)^{-1} f'(v) w$ by $A_{v,k}$. If we can prove that

$$\limsup_{k\to\infty} \sup\{||A_{v,k}||_{\mathcal{L}(X_k^\perp)} \mid v \in N_r(\mathcal{K})\} = 0, \tag{3.8}$$

then the conclusion of this Lemma follows. If (3.8) were not true, we can choose $v_k \in N_r(\mathcal{K})$ and $w_k \in X_k^\perp$ with $||w_k|| = 1$, $k = 1, 2, \cdots$, such that

$$\limsup_{k\to\infty} ||A_{v_k,k} w_k|| > 0. \tag{3.9}$$

Without loss of generality, we assume that $v_k \rightharpoonup v_0$ in X and $w_k \rightharpoonup w_0$ in X as $k \to \infty$. Since for any $2 \leq p < 2^*$, X can be compactly embedded into the radially symmetric L^p space (see, for example, [24, Corollary 1.26])

$$L_r^p(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) \mid u \text{ is radially symmetric}\},$$

combining the condition $(\mathbf{F_1})$, we can get that

$$\lim_{k\to\infty} \sup\{\int_{\mathbb{R}^N} |f'(v_k) w_k h - f'(v_0) w_0 h| \mid h \in X,\ ||h|| \leq 1\} = 0.$$

It follows that

$$\lim_{k\to\infty} ||(-\triangle + 1)^{-1}(f'(v_k) w_k - f'(v_0) w_0)|| = 0. \tag{3.10}$$

By (3.10) and Lemma 3.4, we deduce that $\lim_{k\to\infty} ||A_{v_k,k} w_k|| = 0$. But this contradicts (3.9). $\square$

For $u \in \mathcal{K}$, denote $X_k \oplus Z_u$ by $W_{u,k}$ and let $W_{u,k}^\perp$ be the orthogonal complement space of $W_{u,k}$ in Y. Let $P_{W_{u_i,k}} : Y \to W_{u_i,k}$ and $P_{W_{u_i,k}^\perp} : Y \to W_{u_i,k}^\perp$ be the orthogonal projections.

Lemma 3.6. *Suppose that $\kappa := \max\{\tau_{u_i} \mid 1 \leq i \leq s\}$ is sufficiently small, where τ_{u_i} comes from (3.4). Then there exist $C > 0$ and $l_\kappa \in \mathbb{N}$ such that if $k \geq l_\kappa$ and $v \in B_X(u_i, \tau_{u_i})$ for some $1 \leq i \leq s$, then $P_{W_{u_i,k}^\perp} \nabla^2 J(v)|_{W_{u_i,k}^\perp}$ is invertible and*

$$||(P_{W_{u_i,k}^\perp} \nabla^2 J(v)|_{W_{u_i,k}^\perp})^{-1}||_{\mathcal{L}(W_{u_i,k}^\perp)} \leq C. \tag{3.11}$$

Proof. We note that for $w \in W^{\perp}_{u_i,k}$,

$$P_{W^{\perp}_{u_i,k}} \nabla^2 J(v)w = w - P_{W^{\perp}_{u_i,k}} (-\triangle + 1)^{-1} f'(u)w.$$

Since for any $p \in [2, 2^*)$, X can be compactly embedded into the radially symmetric L^p space, by the condition $(\mathbf{F_1})$, we deduce that $w \mapsto P_{W^{\perp}_{u_i,k}} (-\triangle + 1)^{-1} f'(v)w$ is a compact operator. It follows that $P_{W^{\perp}_{u_i,k}} \nabla^2 J(v)|_{W^{\perp}_{u_i,k}}$ is a Fredholm operator with index zero. Therefore, if we can prove that there exists $C > 0$ which is independent of k such that, for sufficiently large k,

$$||P_{W^{\perp}_{u_i,k}} \nabla^2 J(v)w||_{\mathcal{L}(W^{\perp}_{u_i,k})} \geq \frac{1}{C}||w||, \ \forall w \in W^{\perp}_{u_i,k}, \ \forall v \in B_X(u_i, \tau_{u_i})$$

then the conclusion of this Lemma follows.

Without loss of generality, we assume that $u_i \equiv u_1$ and for the sake of simplicity, we denote the operator $P_{W^{\perp}_{u_1,k}} \nabla^2 J(v)|_{W^{\perp}_{u_1,k}}$ by $H_{v,k}$. If such $C > 0$ does not exist, then there exist sequences $\{\tau^k_{u_1}\}$, $\{v_k\} \subset X$ and $\{w_k\} \subset Y$ such that $\tau^k_{u_1} \to 0$ as $k \to \infty$, $v_k \in B_X(u_1, \tau^k_{u_1})$, $w_k \in W^{\perp}_{u_1,k}$, $||w_k|| = 1$, $k = 1, 2, \cdots$ and

$$\lim_{k\to\infty} ||H_{v_k,k} w_k|| = 0. \tag{3.12}$$

Passing to a subsequence, we may assume that $w_k \rightharpoonup w_0$ in Y as $k \to \infty$. By $\tau^k_{u_1} \to 0$ as $k \to \infty$ and the assumption that $\{v_k\} \subset B_X(u_1, \tau^k_{u_1})$, we get that

$$\lim_{k\to\infty} ||v_k - u_1|| = 0. \tag{3.13}$$

By $w_k \in W^{\perp}_{u_1,k}$ and $w_k \rightharpoonup w_0$ in Y, we get that $w_0 \perp X \oplus Z_{u_1}$. Combining the condition $(\mathbf{F_1})$, (3.13) and the fact that $w_k \rightharpoonup w_0$ in Y leads to

$$\lim_{k\to\infty} ||(-\triangle + 1)^{-1}(f'(v_k)w_k - f'(u_1)w_k)|| = 0 \tag{3.14}$$

and

$$\lim_{k\to\infty} ||(-\triangle + 1)^{-1}(f'(u_1)w_k - f'(u_1)w_0)|| = 0. \tag{3.15}$$

By (3.15) and (3.14), we get that

$$\lim_{k\to\infty} ||(-\triangle + 1)^{-1}(f'(v_k)w_k - f'(u_1)w_0)|| = 0. \tag{3.16}$$

By Lemma 3.4, we deduce that

$$\lim_{k\to\infty} ||P_{W^{\perp}_{u_1,k}} h - P_{(X \oplus Z_{u_1})^{\perp}} h|| = 0, \ \forall h \in Y, \tag{3.17}$$

where $P_{(X\oplus Z_{u_1})^{\perp}} : Y \to (X \oplus Z_{u_1})^{\perp}$ is the orthogonal projection. By (3.16) and (3.17), we get that

$$\lim_{k\to\infty} ||P_{W^{\perp}_{u_1,k}}((-\triangle + 1)^{-1} f'(v_k)w_k) - P_{(X\oplus Z_{u_1})^{\perp}}((-\triangle + 1)^{-1} f'(u_1)w_0)|| = 0. \tag{3.18}$$

By definition,

$$H_{v_k,k} w_k = w_k - P_{W^{\perp}_{u_1,k}}(-\triangle + 1)^{-1} f'(v_k)w_k. \tag{3.19}$$

By (3.18) and the assumption $\lim_{k\to\infty} ||H_{v_k,k} w_k|| = 0$, we deduce that $\{w_k\}$ is compact in Y. Therefore, $||w_k - w_0|| \to 0$ as $k \to \infty$. It follows that $||w_0|| = 1$, since $||w_k|| = 1$ for every k.

Sending k into infinity in the equality (3.19), by $w_0 \in (X \oplus Z_{u_1})^{\perp}$, (3.12) and (3.18), we get that

$$P_{(X\oplus Z_{u_1})^{\perp}}(w_0 - (-\triangle + 1)^{-1} f'(u_1)w_0) = 0. \tag{3.20}$$

By $w_0 \perp X$ and $u_1 \in X$, we have

$$\begin{aligned} & \langle w_0 - (-\triangle + 1)^{-1} f'(u_1) w_0, h \rangle \\ = \quad & \langle w_0, h \rangle - \langle (-\triangle + 1)^{-1} f'(u_1) h, w_0 \rangle = 0, \ \forall h \in X. \end{aligned} \tag{3.21}$$

Since for any $h \in Z_{u_1}$,

$$h - (-\triangle + 1)^{-1} f'(u_1) h = 0,$$

we get that

$$\begin{aligned} & \langle w_0 - (-\triangle + 1)^{-1} f'(u_1) w_0, h \rangle \\ = \quad & \langle h - (-\triangle + 1)^{-1} f'(u_1) h, w_0 \rangle = 0, \ \forall h \in Z_{u_1}. \end{aligned} \tag{3.22}$$

By (3.21) and (3.22), we get that

$$P_{X \oplus Z_{u_1}} (w_0 - (-\triangle + 1)^{-1} f'(u_1) w_0) = 0. \tag{3.23}$$

By (3.20) and (3.23), we obtain

$$w_0 - (-\triangle + 1)^{-1} f'(u_1) w_0 = 0,$$

that is, w_0 is an eigenfunction of (3.2) with $u = u_1 \in \mathcal{K}$. But w_0 satisfies $w_0 \perp X \oplus Z_{u_1}$ and $||w_0|| = 1$. This contradicts Lemma 3.2. $\square$

For $v \in \cup_{i=1}^s B_X(u_i, \tau_{u_i})$, let

$$\mathcal{T}_v = \text{span}\{\sum_{i=1}^s \xi_i(v) \frac{\partial u_i}{\partial x_j} \mid 1 \le j \le N\}. \tag{3.24}$$

The space $X_k \oplus \mathcal{T}_v$ is denoted by $E_{v,k}$. Let $P_{E_{v,k}^\perp} : Y \to E_{v,k}^\perp$ be the orthogonal projection.

Lemma 3.7. *Suppose that $\kappa = \max\{\tau_{u_i} \mid 1 \le i \le s\}$ is sufficiently small. Then there exist $C' > 0$ and $l_\kappa \in \mathbb{N}$ such that if $k \ge l_\kappa$, then for every $v \in \cup_{i=1}^s B_X(u_i, \tau_{u_i})$, the operator $P_{E_{v,k}^\perp} \nabla^2 J(v)|_{E_{v,k}^\perp}$ is invertible and*

$$||(P_{E_{v,k}^\perp} \nabla^2 J(v)|_{E_{v,k}^\perp})^{-1}||_{\mathcal{L}(E_{v,k}^\perp)} \le C'. \tag{3.25}$$

Proof. As the proof of Lemma 3.6, it suffices to prove that there exists $C' > 0$ which is independent of k such that, for sufficiently large k,

$$||P_{E_{v,k}^\perp} \nabla^2 J(v) w||_{\mathcal{L}(E_{v,k}^\perp)} \ge \frac{1}{C'} ||w||, \ \forall w \in E_{v,k}^\perp, \ \forall v \in \cup_{i=1}^s B_X(u_i, \tau_{u_i}). \tag{3.26}$$

Without loss of generality, we assume that $v \in B(u_1, \tau_{u_1})$. Let $P_{X_k} : Y \to X_k$ and $P_{\mathcal{T}_v} : Y \to \mathcal{T}_v$ be orthogonal projections. For $h \in Y$,

$$P_{E_{v,k}^\perp} h = h - P_{X_k} h - P_{\mathcal{T}_v} h, \tag{3.27}$$

and

$$P_{\mathcal{T}_v} h = \sum_{j=1}^N \Big\langle h, \sum_{i=1}^s \xi_i(v) \frac{\partial u_i}{\partial x_j} \Big\rangle \frac{\sum_{i=1}^s \xi_i(v) \frac{\partial u_i}{\partial x_j}}{|| \sum_{i=1}^s \xi_i(v) \frac{\partial u_i}{\partial x_j} ||^2}. \tag{3.28}$$

Since $\{\xi_i \mid 1 \le i \le s\}$ is a partition of unity, we get that for every $1 \le j \le N$,

$$\begin{aligned} || \frac{\partial u_1}{\partial x_j} - \sum_{i=1}^s \xi_i(v) \frac{\partial u_i}{\partial x_j} || \quad &= \quad || \sum_{i=1}^s \xi_i(v) \frac{\partial u_1}{\partial x_j} - \sum_{i=1}^s \xi_i(v) \frac{\partial u_i}{\partial x_j} || \\ &\le \quad \sum_{i=1}^s \xi_i(v) || \frac{\partial u_1}{\partial x_j} - \frac{\partial u_i}{\partial x_j} ||. \end{aligned} \tag{3.29}$$

If $\xi_i(v) \neq 0$, then $v \in B_X(u_i, \tau_{u_i})$. Combining the assumption $v \in B_X(u_1, \tau_{u_1})$, we get that $u_1 \in B_X(u_i, 2\tau_{u_i}) \cap \mathcal{K}$. Therefore, by (3.3), we deduce that

$$\sum_{i=1}^{s} ||\frac{\partial u_1}{\partial x_j} - \frac{\partial u_i}{\partial x_j}|| < \varsigma, \text{ if } \xi_i(v) \neq 0. \tag{3.30}$$

Combining (3.29) and (3.30) leads to

$$||\frac{\partial u_1}{\partial x_j} - \sum_{i=1}^{s} \xi_i(v) \frac{\partial u_i}{\partial x_j}|| < \varsigma, \text{ for every } 1 \leq j \leq N. \tag{3.31}$$

Thus, there exists $C > 0$ which is independent of k such that

$$||P_{\mathcal{T}_v} h - P_{Z_{u_1}} h|| \leq C\varsigma ||h||, \ \forall h \in Y, \tag{3.32}$$

where

$$P_{Z_{u_1}} : Y \to Z_{u_1}, \ h \mapsto \sum_{j=1}^{N} \left\langle h, \frac{\partial u_1}{\partial x_j} \right\rangle \frac{\frac{\partial u_1}{\partial x_j}}{||\frac{\partial u_1}{\partial x_j}||^2}$$

is orthogonal projection. By (3.27) and (3.32), we have

$$||P_{E_{v,k}^{\perp}} h - P_{W_{u_1,k}^{\perp}} h|| \leq C\varsigma ||h||, \ \forall h \in Y. \tag{3.33}$$

For $w \in E_{v,k}^{\perp}$, we have

$$\begin{aligned}
& ||P_{E_{v,k}^{\perp}} \nabla^2 J(v) w|| \qquad (3.34) \\
\geq \ & ||P_{W_{u_1,k}^{\perp}} \nabla^2 J(v) w|| - ||(P_{E_{v,k}^{\perp}} - P_{W_{u_1,k}^{\perp}}) \nabla^2 J(v) w|| \\
\geq \ & ||P_{W_{u_1,k}^{\perp}} \nabla^2 J(v) w|| - C\varsigma ||\nabla^2 J(v)||_{\mathcal{L}(Y)} ||w|| \text{ (by (3.33))} \\
\geq \ & ||P_{W_{u_1,k}^{\perp}} \nabla^2 J(v)(w - P_{Z_{u_1}} w)|| - ||P_{W_{u_1,k}^{\perp}} \nabla^2 J(v)(P_{Z_{u_1}} w)|| - C\varsigma ||\nabla^2 J(v)||_{\mathcal{L}(Y)} ||w|| \\
\geq \ & C||w - P_{Z_{u_1}} w|| - ||\nabla^2 J(v)||_{\mathcal{L}(Y)} ||P_{Z_{u_1}} w|| \\
& -C\varsigma ||\nabla^2 J(v)||_{\mathcal{L}(Y)} ||w|| \text{ (by } w - P_{Z_{u_1}} w \in W_{u_1,k}^{\perp} \text{ and (3.11))} \\
\geq \ & C||w|| - (C + ||\nabla^2 J(v)||_{\mathcal{L}(Y)}) ||P_{Z_{u_1}} w|| - C\varsigma ||\nabla^2 J(v)||_{\mathcal{L}(Y)} ||w|| \\
= \ & C||w|| - (C + ||\nabla^2 J(v)||_{\mathcal{L}(Y)}) ||P_{\mathcal{T}_v} w - P_{Z_{u_1}} w|| \\
& -C\varsigma ||\nabla^2 J(v)||_{\mathcal{L}(Y)} ||w|| \text{ (since } P_{\mathcal{T}_v} w = 0) \\
\geq \ & C||w|| - \varsigma C(C + ||\nabla^2 J(v)||_{\mathcal{L}(Y)}) ||w|| - C\varsigma ||\nabla^2 J(v)||_{\mathcal{L}(Y)} ||w||. \text{ (by (3.32))}
\end{aligned}$$

It follows that if $\kappa > 0$ is sufficiently small, then there exist $l_\kappa \in \mathbb{N}$ and $C' > 0$ such that for every $k \geq l_\kappa$, (3.26) holds. □

Recall that $X_k^{\perp}$ is the orthogonal complement space of X_k in X and $P_k : X \to X_k$, $P_k^{\perp} : X \to X_k^{\perp}$ are orthogonal projections. Let

$$\mathcal{N}_{\delta,\tau,k} := \{u + v \in X \mid u \in X_k, \ \text{dist}_X(u, P_k\mathcal{K}) < \delta, \ v \in X_k^{\perp}, \ ||v|| < \tau\},$$

where $P_k\mathcal{K} = \{P_k v \mid v \in \mathcal{K}\}$. By Lemma 3.4 and the fact that $\mathcal{K}$ is a compact subset of X, we get that as $k \to \infty$, the Hausdorff distance of $\mathcal{K}$ and $P_k\mathcal{K}$,

$$\sup_{v \in P_k\mathcal{K}} \text{dist}_X(v, \mathcal{K}) + \sup_{u \in \mathcal{K}} \text{dist}_X(u, P_k\mathcal{K}) \to 0. \tag{3.35}$$

Thus, for any $\delta > 0$, $\tau > 0$ and $0 < r < \min\{\delta, \tau\}$, if k is sufficiently large, then

$$N_r(\mathcal{K}) \subset \mathcal{N}_{\delta,\tau,k}, \tag{3.36}$$

where $N_r(\mathcal{K})$ comes from (2.8). And for any $r > 0$, if $\delta, \tau \in (0, r/2)$, then for sufficiently large k,

$$\mathcal{N}_{\delta,\tau,k} \subset N_r(\mathcal{K}). \tag{3.37}$$

Let

$$\mathcal{N}_{\delta,k} := \{u \in X_k \mid \text{dist}_X(u, P_k\mathcal{K}) < \delta\}. \tag{3.38}$$

Lemma 3.8. *If $\delta > 0$ is sufficient small and k is sufficiently large, then there exists a C^1-mapping*

$$\pi_k : \mathcal{N}_{\delta,k} \to X_k^{\perp},$$

satisfying

(i) $\langle \nabla I(v + \pi_k(v)), \phi \rangle = 0, \forall \phi \in X_k^{\perp}$;

(ii) $\lim_{k\to\infty} \sup\{||\pi_k(v)|| \mid v \in \mathcal{N}_{\delta,k}\} = 0$;

(iii) $\lim_{k\to\infty} \sup\{||D\pi_k(v)h|| \mid v \in \mathcal{N}_{\delta,k},\ h \in X_k,\ ||h|| = 1\} = 0$;

(iv) *If v is a critical point of $I(v + \pi_k(v))$, then $v + \pi_k(v)$ is a critical point of I.*

Proof. By Lemma 3.5, if $r > 0$ is small enough, then the operator

$$L_{v,k} := P_k^{\perp}\nabla^2 I(v)|_{X_k^{\perp}} : X_k^{\perp} \to X_k^{\perp}$$

is invertible and if $k \geq l_\kappa$,

$$||L_{v,k}^{-1}||_{\mathcal{L}(X_k^{\perp})} \leq 2,\ \forall v \in N_r(\mathcal{K}). \tag{3.39}$$

Assume that $0 < \delta < r$, by (3.37), if k is large enough, then $\mathcal{N}_{\delta,k} \subset N_r(\mathcal{K})$.
For $\rho > 0$ and $v \in \mathcal{N}_{\delta,k}$, define

$$\Psi_{v,k} : \overline{B_{X_k^{\perp}}(0,\rho)} \to X_k^{\perp},\ w \mapsto w - L_{v,k}^{-1}P_k^{\perp}\nabla I(v + w).$$

For any $w_i \in \overline{B_{X_k^{\perp}}(0,\rho)}$, $i = 1, 2$, by the definition of $L_{v,k}$, we have $w_2 - w_1 - L_{v,k}^{-1}P_k^{\perp}\nabla^2 I(v)(w_2 - w_1) = 0$. Therefore,

$$\begin{aligned}
& ||\Psi_{v,k}(w_2) - \Psi_{v,k}(w_1)|| \\
= \ & ||w_2 - w_1 - L_{v,k}^{-1}P_k^{\perp}\nabla^2 I(v + \theta w_2 + (1-\theta)w_1)(w_2 - w_1)|| \\
& \text{(by the mean value theorem, } 0 < \theta = \theta(x) < 1) \\
\leq \ & ||w_2 - w_1 - L_{v,k}^{-1}P_k^{\perp}\nabla^2 I(v)(w_2 - w_1)|| \\
& + ||L_{v,k}^{-1}P_k^{\perp}(\nabla^2 I(v + \theta w_2 + (1-\theta)w_1) - \nabla^2 I(v))(w_2 - w_1)|| \\
= \ & ||L_{v,k}^{-1}P_k^{\perp}(\nabla^2 I(v + \theta w_2 + (1-\theta)w_1) - \nabla^2 I(v))(w_2 - w_1)|| \\
\leq \ & 2||(\nabla^2 I(v + \theta w_2 + (1-\theta)w_1) - \nabla^2 I(v))(w_2 - w_1)|| \text{ (by (3.39))}.
\end{aligned} \tag{3.40}$$

Since $I \in C^2(X, \mathbb{R})$ and $\mathcal{K}$ is compact in X, if δ and ρ are small enough, then for any $v \in \mathcal{N}_{\delta,k}$ and $w \in \overline{B_{X_k^{\perp}}(0,\rho)}$,

$$||\nabla^2 I(v + w) - \nabla^2 I(v)||_{\mathcal{L}(X)} < 1/4.$$

Thus, by (3.40), we get that for any $w_i \in \overline{B_{X_k^{\perp}}(0,\rho)}$, $i = 1, 2$,

$$||\Psi_{v,k}(w_2) - \Psi_{v,k}(w_1)|| \leq \frac{1}{2}||w_2 - w_1||. \tag{3.41}$$

If $\delta > 0$ is small enough and k is large enough, then for every $v \in \mathcal{N}_{\delta,k}$,

$$||\Psi_{v,k}(0)|| \leq \rho/2.$$

Then by (3.41), we get that for every $w \in \overline{B_{X_k^\perp}(0,\rho)}$,

$$||\Psi_{v,k}(w)|| \le ||\Psi_{v,k}(w) - \Psi_{v,k}(0)|| + ||\Psi_{v,k}(0)|| \le \rho. \tag{3.42}$$

By (3.41) and (3.42), $\Psi_{v,k}$ is a contractive mapping in $\overline{B_{X_k^\perp}(0,\rho)}$ if δ and ρ are small enough and k is large enough. Thus, by Banach fixed point theorem, there exists unique fixed point $\pi_k(v) \in \overline{B_{X_k^\perp}(0,\rho)}$. It is easy to verify that π_k is a C^1-mapping and it satisfies the result (**i**).

Now, we give the proof of (**ii**). By $P_k^\perp \nabla I(v+\pi_k(v)) = 0$ and $\pi_k(v) \in X_k^\perp$, we get that

$$\begin{aligned} 0 &= \langle \nabla I(v+\pi_k(v)), \pi_k(v)\rangle \\ &= ||\pi_k(v)||^2 - \int_{\mathbb{R}^N} f(v+\pi_k(v))\cdot \pi_k(v). \end{aligned} \tag{3.43}$$

By Lemma 3.4, we deduce that for any sequence $\{v_k\}$ with $v_k \in \mathcal{N}_{\delta,k}$, $\pi_k(v_k) \rightharpoonup 0$ in X as $k \to \infty$. Combining the compact embedding $X \hookrightarrow L_r^p(\mathbb{R}^N)$, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |f(v_k+\pi_k(v_k))| \cdot |\pi_k(v_k)| = 0.$$

It follows that

$$\lim_{k\to\infty} \sup\{\int_{\mathbb{R}^N} f(v+\pi_k(v))\cdot \pi_k(v) \mid v \in \mathcal{N}_{\delta,k}\} = 0. \tag{3.44}$$

The conclusion (**ii**) follows from (3.43) and (3.44).

Differentiating equation $P_k^\perp \nabla I(v+\pi_k(v)) = 0$ for the variable v in the direction $h \in X_k$, we get that

$$D\pi_k(v)h - P_k^\perp(-\triangle+1)^{-1} f'(v+\pi_k(v))(h + D\pi_k(v)h) = 0. \tag{3.45}$$

Note that $D\pi_k(v)h \in X_k^\perp$. By (3.39), (3.45) and $\lim_{k\to\infty} ||\pi_k(v)|| = 0$, we get that if k is large enough, then

$$\begin{aligned} \frac{1}{2}||D\pi_k(v)h|| &\le ||D\pi_k(v)h - P_k^\perp(-\triangle+1)^{-1} f'(v+\pi_k(v))D\pi_k(v)h|| \\ &= ||P_k^\perp(-\triangle+1)^{-1} f'(v+\pi_k(v))h|| \end{aligned} \tag{3.46}$$

It follows that for sufficiently large k,

$$\sup\{||D\pi_k(v)h|| \mid v \in \mathcal{N}_{\delta,k},\ h \in X_k,\ ||h|| \le 1\} < \infty. \tag{3.47}$$

By (3.45), we get that

$$||D\pi_k(v)h||^2 = \int_{\mathbb{R}^N} f'(v+\pi_k(v))\cdot(h + D\pi_k(v)h)\cdot D\pi_k(v)h. \tag{3.48}$$

(3.47) and the same argument as (3.44) yield

$$\lim_{k\to\infty} \sup\{\int_{\mathbb{R}^N} f'(v+\pi_k(v))\cdot(h+D\pi_k(v)h)\cdot D\pi_k(v)h \mid v \in \mathcal{N}_{\delta,k},\ h \in X_k,\ ||h|| \le 1\} = 0.$$

Combining (3.48), we get the conclusion (**iii**).

By (**iii**), if k is sufficiently large, then

$$\{h + D\pi_k(v)h \mid h \in X_k\} + X_k^\perp = X.$$

Combining the result (**i**), we get that if v_0 is a critical point of $I(v+\pi_k(v))$, then $v_0 + \pi_k(v_0)$ is a critical point of I. □

Remark 3.9. *By* $(\mathbf{ii})$ *and* $(\mathbf{iv})$ *of Lemma* 3.8, $\mathcal{N}_{\delta,\tau,k}$ *is a neighborhood of* $\mathcal{K}$ *if*

$$\tau > \sup\{||\pi_k(v)|| \mid v \in \mathcal{N}_{\delta,k}\}. \tag{3.49}$$

Lemma 3.10. *Let* $\mathcal{I}_k(u) = \frac{1}{2}||P_k^{\perp}u||^2 + I(P_k u + \pi_k(P_k u))$. *Then*

$$\lim_{k\to\infty} ||\mathcal{I}_k - I||_{C^1(\overline{\mathcal{N}_{\delta,\tau,k}})} = 0.$$

Proof. By definition, we have

$$\mathcal{I}_k(u) = \frac{1}{2}||u||^2 + \frac{1}{2}||\pi_k(P_k u)||^2 - \int_{\mathbb{R}^N} F(P_k u + \pi_k(P_k u)).$$

For any sequence $\{u_k\}$ with $u_k \in \overline{\mathcal{N}_{\delta,\tau,k}}$, by the mean value theorem, we get that

$$\begin{aligned} F(P_k u_k + \pi_k(P_k u_k)) - F(u_k) &= \zeta(u_k,\theta)(P_k u_k + \pi_k(P_k u_k) - u_k) \\ &= \zeta(u_k,\theta)(\pi_k(P_k u_k) - P_k^{\perp} u_k) \end{aligned}$$

where

$$\zeta(u_k,\theta) = f'(\theta P_k u_k + \theta\pi_k(P_k u_k) + (1-\theta)u_k)$$

with $0 < \theta(x) < 1$, $x \in \mathbb{R}^N$. Then we have

$$\int_{\mathbb{R}^N} \Big|F(P_k u_k + \pi_k(P_k u_k)) - F(u_k)\Big| = \int_{\mathbb{R}^N} |\zeta(u_k,\theta)| \cdot |\pi_k(P_k u_k) - P_k^{\perp} u_k|. \tag{3.50}$$

By $(\mathbf{ii})$ of Lemma 3.8, we get that for every $2 \le p < 2^*$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |\pi_k(P_k u_k)|^p = 0. \tag{3.51}$$

By Lemma 3.4,we have

$$P_k^{\perp} u_k \rightharpoonup 0 \text{ in } X. \tag{3.52}$$

Since X can be compactly embedded into $L^p_r(\mathbb{R}^N)$, by (3.52), we get that for every $2 \le p < 2^*$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |P_k^{\perp} u_k|^p = 0. \tag{3.53}$$

By (3.50), (3.51), (3.53) and the condition $(\mathbf{F_1})$, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \Big|F(P_k u_k + \pi_k(P_k u_k)) - F(u_k)\Big| = 0.$$

Thus

$$\lim_{k\to\infty} \sup\{\int_{\mathbb{R}^N} \Big|F(P_k u + \pi_k(P_k u)) - F(u)\Big| \mid u \in \overline{\mathcal{N}_{\delta,\tau,k}}\} = 0. \tag{3.54}$$

By $(\mathbf{ii})$ of Lemma 3.8 and (3.54), we get that

$$\lim_{k\to\infty} ||\mathcal{I}_k - I||_{C^0(\overline{\mathcal{N}_{\delta,\tau,k}})} = 0. \tag{3.55}$$

For $h \in X$,

$$\begin{aligned} \langle \nabla\mathcal{I}_k(u), h\rangle &= \langle u, h\rangle + \langle \pi_k(P_k u), D\pi_k(P_k u)(P_k h)\rangle \\ &\quad - \int_{\mathbb{R}^N} f(P_k u + \pi_k(P_k u)) \cdot (P_k h + D\pi_k(P_k u)(P_k h)). \end{aligned}$$

By $(\mathbf{iii})$ of Lemma 3.8 and the same argument as above, we can get that

$$\lim_{k\to\infty} \sup\{\langle \nabla\mathcal{I}_k(u) - \nabla I(u), h\rangle \mid u \in \overline{\mathcal{N}_{\delta,\tau,k}},\ ||h|| \le 1\} = 0. \tag{3.56}$$

The result of this Lemma follows from (3.55) and (3.56). $\square$

Remark 3.11. *For $r > 0$, let $\sigma \in (0, \sigma_{r/2})$, where $\sigma_{r/2}$ comes from Lemma 2.7, and let $a \in (c - \sigma, c)$, $b \in (c, c + \sigma)$ be regular values of I, where c comes from (2.3). By Lemma 2.7, there exists a GM pair (W, W_-) of $\mathcal{K}_a^b$ associated with some pseudo-gradient vector field of I such that $W \subset N_{r/2}(\mathcal{K}_a^b)$. By (3.36), if $0 < r < \min\{\delta, \tau\}$, then $N_r(\mathcal{K}) \subset \mathcal{N}_{\delta,\tau,k}$ if k is sufficiently large. Denote the critical set of $\mathcal{I}_k$ in $\mathcal{N}_{\delta,\tau,k}$ by $\widehat{\mathcal{K}}_k$. By $(\mathbf{i})$ and $(\mathbf{iv})$ of Lemma 3.8, we deduce that $\widehat{\mathcal{K}}_k = P_k\mathcal{K}_a^b$. Then by (3.35), $\widehat{\mathcal{K}}_k \subset \operatorname{int} W$ if k is large enough. By [9, Theorem III.4] and Lemma 3.10, we infer that for sufficiently large k, (W, W_-) is also a GM pair of $\mathcal{I}_k$ for $\widehat{\mathcal{K}}_k$ associated with some pseudo-gradient vector filed of $\mathcal{I}_k$.*

For $v \in \mathcal{N}_{\delta,k}$, denote $I(v + \pi_k(v))$ by $g_k(v)$. And denote the critical set of g_k in W by $\mathcal{K}_k$. By $(\mathbf{i})$ and $(\mathbf{iv})$ of Lemma 3.8, we deduce that $\mathcal{K}_k = P_k\mathcal{K}_a^b = \widehat{\mathcal{K}}_k$. Let (W_k, W_k^-) be a GM pair of g_k for $\mathcal{K}_k$. Note that for $u = w + v \in \mathcal{N}_{\delta,\tau,k}$ with $w \in X_k^\perp$, $v \in X_k$, $\mathcal{I}_k(u) = \frac{1}{2}||w||^2 + g_k(v)$. By shifting theorem (see Lemma 5.1 of [7]), we have

$$\check{H}^q(W_k, W_k^-) = \check{H}^q(W, W^-),\ q = 0, 1, 2, \cdots.$$

Combining Lemma 2.5, we get that, for sufficiently large k,

$$\check{H}^1(W_k, W_k^-) = \check{H}^1(W, W^-) \neq 0. \tag{3.57}$$

4 A variational reduction for the functional E_ϵ

For $v \in \cup_{i=1}^s B_X(u_i, \tau_{u_i})$ and $y \in \mathbb{R}^N$, denote the space

$$\{\zeta(\cdot - y) \mid \zeta \in X_k\} \oplus \mathcal{T}_v(\cdot - y)$$

by $T_{v,y,k}$, where $\mathcal{T}_v$ comes from (3.24). Denote the orthogonal complemental space of $T_{v,y,k}$ in Y by $T_{v,y,k}^\perp$.

Recall that (see (3.38))

$$\mathcal{N}_{\delta,k} = \{u \in X_k \mid \text{dist}_X(u, P_k\mathcal{K}) < \delta\}.$$

For $v \in \mathcal{N}_{\delta,k}$, define

$$L_{v,y,\epsilon,k} : T_{v,y,k}^\perp \to T_{v,y,k}^\perp$$

by

$$w \in T_{v,y,k}^\perp \mapsto w - S_{v,y,k}(-\triangle + 1 + V(\epsilon x))^{-1}(f'(v(\cdot - y))w) \tag{4.1}$$

where $S_{v,y,k} : Y \to T_{v,y,k}^\perp$ is orthogonal projection and the operator $(-\triangle + 1 + V(\epsilon x))^{-1}$ is defined by (2.2).

Lemma 4.1. *Given $R > 0$, there exist $\delta_0 > 0$, $\epsilon_0 > 0$, $l^* > 0$ and $C > 0$ which are independent of k, such that if $k \geq l^*$, $0 < \delta \leq \delta_0$ and $0 \leq \epsilon \leq \epsilon_0$, then for any $v \in \overline{\mathcal{N}_{\delta,k}}$ and $y \in \overline{B_{\mathbb{R}^N}(0, R)}$, $L_{v,y,\epsilon,k}$ is invertible and*

$$||L_{v,y,\epsilon,k}w|| \geq C||w||,\ \forall |y| \leq R,\ \forall w \in T_{v,y,k}^\perp. \tag{4.2}$$

Proof. Suppose $\kappa = \max\{\tau_{u_i} \mid 1 \leq i \leq s\}$ is small enough such that Lemma 3.7 holds. By (3.37), for sufficiently small $\delta_0 > 0$, there exists $l'_\kappa > 0$ such that $\mathcal{N}_{\delta_0,k} \subset \cup_{i=1}^s B_X(u_i, \tau_{u_i})$ if $k \geq l'_\kappa$. Note that $L_{v,0,0,k}$ is exactly the operator $P_{E_{v,k}^\perp}\nabla^2 J(v)|_{E_{v,k}^\perp}$ which has been defined in Lemma 3.7 and for every $w \in T_{v,y,k}^\perp$,

$$L_{v,y,0,k}w = L_{v,0,0,k}w(\cdot - y).$$

Thus, by Lemma 3.7, there exists $C' > 0$ such that if $k \geq l^* := \max\{l_\kappa, l'_\kappa\}$, then for any $v \in \mathcal{N}_{\delta_0,k}$,

$$||L_{v,y,0,k}w|| \geq C'||w||,\ \forall |y| \leq R,\ \forall w \in T_{v,y,k}^\perp,$$

where l_κ is the constant comes from Lemma 3.7. Therefore, to prove (4.2), it suffices to prove that

$$\lim_{\epsilon\to 0}\sup\Big\{||L_{v,y,\epsilon,k}w - L_{v,y,0,k}w|| \mid w\in T^{\perp}_{v,y,k},\ ||w||\le 1, \quad v\in\overline{\mathcal{N}_{\delta_0,k}},\ y\in\overline{B_{\mathbb{R}^N}(0,R)},\ k\ge l^*\Big\}=0. \tag{4.3}$$

If we can prove that for any given sequences $\{k_n\}\subset\mathbb{N}$, $\{\epsilon_n\}\subset(0,+\infty)$, $\{y_n\}\subset\overline{B_{\mathbb{R}^N}(0,R)}$, $\{v_n\}$ and $\{w_n\}$ which satisfy that $\epsilon_n\to 0$ as $n\to\infty$, $v_n\in\overline{\mathcal{N}_{\delta_0,k_n}}$, $w_n\in T^{\perp}_{v_n,y_n,k_n}$ and $||w_n||\le 1$, $n=1,2,\cdots$,

$$\lim_{n\to\infty}||L_{v_n,y_n,\epsilon_n,k_n}w_n - L_{v_n,y_n,0,k_n}w_n|| = 0, \tag{4.4}$$

then (4.3) holds. We only give the proof of (4.4) in the case $k_n\to\infty$, $n\to\infty$, since the proofs in other cases are similar. Without loss of generality, we assume that $\{k_n\}$ is exactly the sequence $\{k\}$ and we shall denote ϵ_n, y_n, v_n and w_n by ϵ_k, y_k, v_k and w_k respectively, $k=1,2,\cdots$.

Passing to a subsequence, we may assume that as $k\to\infty$, $y_k\to y_0$, $v_k\rightharpoonup v_0$ in X and $w_k\rightharpoonup w_0$ in Y.

Let

$$\eta_k = (-\triangle + 1 + V(\epsilon_k x))^{-1}(f'(v_k(\cdot - y_k))w_k).$$

It is easy to verify that $\{\eta_k\}$ is bounded in Y and

$$\eta_k = (-\triangle+1)^{-1}(f'(v_k(\cdot-y_k))w_k) - (-\triangle+1)^{-1}V(\epsilon_k)\eta_k. \tag{4.5}$$

Passing to a subsequence, we may assume that $\eta_k\rightharpoonup\eta_0$ in Y as $k\to\infty$.

By definition of $L_{v,y,\epsilon,k}$ and (4.5), we get that

$$L_{v_k,y_k,\epsilon,k}w - L_{v_k,y_k,0,k}w = S_{v_k,y_k,k}(-\triangle+1)^{-1}V(\epsilon_k x)\eta_k. \tag{4.6}$$

The condition $(\mathbf{V_1})$ implies that $V(0)=0$. It follows that for any $h\in Y$,

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} V(\epsilon_k x)\eta_k h = 0. \tag{4.7}$$

Since η_k is a weak solution of the equation:

$$-\triangle\eta_k + \eta_k + V(\epsilon_k x)\eta_k = f'(v_k(\cdot - y_k))w_k, \tag{4.8}$$

by (4.7), $y_k\to y_0$, $\eta_k\rightharpoonup\eta_0$ and $w_k\rightharpoonup w_0$ in Y, we get that η_0 is a weak solution of the equation:

$$-\triangle\eta_0 + \eta_0 = f'(v_0(\cdot - y_0))w_0. \tag{4.9}$$

From (4.8) and (4.9), we obtain

$$\begin{aligned} & -\triangle(\eta_k-\eta_0) + (\eta_k-\eta_0) + V(\epsilon_k x)(\eta_k-\eta_0) \\ = \ & (f'(v_k(\cdot-y_k))w_k - f'(v_0(\cdot-y_0))w_0) - V(\epsilon_k x)\eta_0. \end{aligned}$$

Multiplying the above equation by $\eta_k-\eta_0$ and integrating, we get that there exists a constant $C>0$ such that

$$\begin{aligned} & C||\eta_k-\eta_0||^2 \\ \le\ & ||\eta_k-\eta_0||^2 + \int_{\mathbb{R}^N} V(\epsilon_k x)(\eta_k-\eta_0)^2 \text{ (by the condition } (\mathbf{V_0})) \\ =\ & \int_{\mathbb{R}^N}\Big(f'(v_k(\cdot-y_k))w_k - f'(v_0(\cdot-y_0))w_0 - V(\epsilon_k x)\eta_0\Big)\cdot(\eta_k-\eta_0) \\ \le\ & \int_{\mathbb{R}^N}\Big|f'(v_k(\cdot-y_k))w_k - f'(v_0(\cdot-y_0))w_0\Big|\cdot|\eta_k-\eta_0| \\ & +(\int_{\mathbb{R}^N} V^2(\epsilon_k x)\eta_0^2)^{\frac{1}{2}}\cdot||\eta_k-\eta_0||_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{4.10}$$

Since $v_k \rightharpoonup v_0$ in X and $y_k \to y_0$ as $k \to \infty$, by the fact that X can be compactly embedding into $L_r^p(\mathbb{R}^N)$ ($\forall p \in [2, 2^*)$), we get that

$$\lim_{k\to\infty} ||v_k(\cdot - y_k) - v_0(\cdot - y_0)||_{L^p(\mathbb{R}^N)} = 0,\ \forall p \in [2, 2^*). \tag{4.11}$$

By (4.11) and the condition $(\mathbf{F_1})$, we get that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left| f'(v_k(\cdot - y_k))w_k - f'(v_0(\cdot - y_0))w_0 \right| \cdot |\eta_k - \eta_0| = 0. \tag{4.12}$$

By (4.10), (4.12) and

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} V^2(\epsilon_k x)\eta_0^2 = 0, \tag{4.13}$$

we get that

$$\lim_{k\to\infty} ||\eta_k - \eta_0|| = 0. \tag{4.14}$$

(4.13) and (4.14) yield

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} V^2(\epsilon_k x)\eta_k^2 = 0. \tag{4.15}$$

It follows that

$$\lim_{k\to\infty} ||(-\triangle + 1)^{-1} V(\epsilon_k x)\eta_k|| = 0. \tag{4.16}$$

Combining (4.16) and (4.6) leads to (4.4).

Finally, by definition, $L_{v,y,\epsilon,k}$ is a Fredholm operator with index zero and by (4.2), it is an injection. Therefore, it is invertible. □

Theorem 4.2. *Given $R > 0$. There exist $\delta^* > 0$ and $\epsilon^* > 0$ such that if $0 < \delta \le \delta^*$ and $0 \le \epsilon \le \epsilon^*$, then there exist $k(\delta)$ and a C^1-mapping*

$$w_{\delta,k}(\cdot,\cdot,\epsilon) : \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)} \to Y,\ (u,y) \mapsto w_{\delta,k}(u,y,\epsilon)$$

for $k \ge k(\delta)$, satisfying

- **(i)** $w_{\delta,k}(u,y,\epsilon) \in T^{\perp}_{u,y,k}$, $\forall (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}$;
- **(ii)** $\langle \nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)), \phi\rangle = 0, \forall \phi \in T^{\perp}_{u,y,k}$;
- **(iii)** $w_{\delta,k}(u,y,0) = (\pi_k(u))(\cdot - y), \forall (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}$*;*
- **(iv)** *for any $r > 0$, there exists $\delta_r > 0$ such that if $0 < \delta \le \delta_r$, $u \in \overline{\mathcal{N}_{\delta,k}}$, $y \in \overline{B_{\mathbb{R}^N}(0,R)}$ and $k \ge k(\delta)$, then $||w_{\delta,k}(u,y,\epsilon)|| \le r$;*
- **(v)** *for any $n > 0$,*

$$\begin{aligned}&\sup\{||(1+|x|)^n w_{\delta,k}(u,y,\epsilon)||_{L^\infty(\mathbb{R}^N)} \mid (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)},\ 0 \le \epsilon \le \epsilon^*\} \\ &< \infty.\end{aligned} \tag{4.17}$$

Proof. By Lemma 4.1, we know that for any $R > 0$, $L_{u,y,\epsilon,k}$ is invertible if $0 < \delta \le \delta_0$, $0 \le \epsilon \le \epsilon_0$ and $k \ge l^*$. Moreover, the upper bound of $||L^{-1}_{u,y,\epsilon,k}||$ is independent of u, y, ϵ and k. For $u \in \overline{\mathcal{N}_{\delta,k}}$ and $r > 0$, let

$$\Phi_{u,y,\epsilon,k} : \overline{B_{T^{\perp}_{u,y,k}}(0,r)} \to T^{\perp}_{u,y,k},$$

$$w \mapsto w - L^{-1}_{u,y,\epsilon,k} S_{u,y,k} \nabla E_\epsilon(u(\cdot - y) + w).$$

Now, we show that if r, δ and ϵ are small enough and k is large enough, then for any $u \in \overline{\mathcal{N}_{\delta,k}}$, $\Phi_{u,y,\epsilon,k}$ is a contractive mapping in $\overline{B_{T^{\perp}_{u,y,k}}(0,r)}$.

Using

$$\begin{aligned} & \nabla E_\epsilon(u(\cdot - y) + w) \\ = \quad & u(\cdot - y) + w - (-\triangle + 1 + V(\epsilon x))^{-1} f(u(\cdot - y) + w) \end{aligned}$$

and the mean value theorem, we get that for any $w_1, w_2 \in \overline{B_{T^{\perp}_{u,y,k}}(0,r)}$, $\Phi_{u,y,\epsilon,k}(w_1) - \Phi_{u,y,\epsilon,k}(w_2)$ equals

$$\begin{aligned} & (w_1 - w_2) - L^{-1}_{u,y,\epsilon,k} S_{u,y,k} \Big\{ (w_1 - w_2) \\ & -(-\triangle + 1 + V(\epsilon x))^{-1} (f'(u(\cdot - y) + \tilde{w}) \cdot (w_1 - w_2)) \Big\} \\ = \quad & (w_1 - w_2) - L^{-1}_{u,y,\epsilon,k} S_{u,y,k} \Big\{ (w_1 - w_2) \\ & -(-\triangle + 1 + V(\epsilon x))^{-1} f'(u(\cdot - y))(w_1 - w_2) \\ & -(-\triangle + 1 + V(\epsilon x))^{-1} (f'(u(\cdot - y) + \tilde{w}) - f'(u(\cdot - y)))(w_1 - w_2) \Big\} \qquad (4.18) \end{aligned}$$

where $\tilde{w} = \theta w_1 + (1-\theta) w_2$ for some $0 < \theta < 1$. By the condition $(\mathbf{F_1})$, we can prove that

$$\begin{aligned} \lim_{r \to 0} \sup \{ & ||(-\triangle + 1 + V(\epsilon x))^{-1} (f'(u(\cdot - y) + \tilde{w}) - f'(u(\cdot - y)))\varphi|| \qquad (4.19) \\ & | \; u \in \overline{\mathcal{N}_{\delta,k}}, \; |y| \le R, \; \varphi \in Y, \; ||\varphi|| \le 1, \; 0 \le \epsilon \le \epsilon_0 \} = 0. \end{aligned}$$

By $||L^{-1}_{u,y,\epsilon,k}||_{\mathcal{L}(Y)} \le 1/C$ (see Lemma 4.1), $||S_{u,y,k}||_{\mathcal{L}(Y)} \le 1$ and (4.19), we deduce that if r is small enough, then

$$\begin{aligned} & ||L^{-1}_{u,y,\epsilon,k} S_{u,y,k} (-\triangle + 1 + V(\epsilon x))^{-1} (f'(u(\cdot - y) + \tilde{w}) - f'(u(\cdot - y)))(w_1 - w_2)|| \qquad (4.20) \\ & \le \frac{1}{C} ||(-\triangle + 1 + V(\epsilon x))^{-1} (f'(u(\cdot - y) + \tilde{w}) - f'(u(\cdot - y)))(w_1 - w_2)|| \\ & \le \frac{1}{2} ||w_1 - w_2||. \end{aligned}$$

By the definition of $L_{u,y,\epsilon,k}$,

$$\begin{aligned} & L^{-1}_{u,y,\epsilon,k} S_{u,y,k} \Big\{ (w_1 - w_2) - (-\triangle + 1 + V(\epsilon x))^{-1} (f'(u(\cdot - y))(w_1 - w_2)) \Big\} \qquad (4.21) \\ & = (w_1 - w_2). \end{aligned}$$

Combining (4.20), (4.21) and (4.18), we deduce that there exists $r_0 > 0$ such that if $0 < r \le r_0$, $0 < \delta \le \delta_0$, $0 \le \epsilon \le \epsilon_0$ and $k \ge l^*$, then for any $(u, y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}$ and $w_1, w_2 \in \overline{B_{T^{\perp}_{u,y,k}}(0,r)}$,

$$||\Phi_{u,y,\epsilon,k}(w_1) - \Phi_{u,y,\epsilon,k}(w_2)|| \le \frac{1}{2} ||w_1 - w_2||. \qquad (4.22)$$

Claim: For any $0 < r \le r_0$, there exist ϵ_r, δ_r and $k(\delta, r)$ such that if $0 < \delta \le \delta_r$, $0 \le \epsilon \le \epsilon_r$ and $k \ge k(\delta, r)$, then

$$||\Phi_{u,y,\epsilon,k}(0)|| \le r/2, \; \forall (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}. \qquad (4.23)$$

Let $h_{u,y,\epsilon} = (-\triangle + 1 + V(\epsilon x))^{-1} f(u(\cdot - y))$. It is easy to verify

$$h_{u,y,\epsilon} = (-\triangle + 1)^{-1} f(u(\cdot - y)) - (-\triangle + 1)^{-1} V(\epsilon x) h_{u,y,\epsilon}. \qquad (4.24)$$

The same argument as (4.15) yields

$$\lim_{\epsilon \to 0} \sup \{ \int_{\mathbb{R}^N} V^2(\epsilon x) h^2_{u,y,\epsilon} \; | \; u \in \overline{\mathcal{N}_{\delta_0,k}}, \; y \in \overline{B_{\mathbb{R}^N}(0,R)}, \; k \ge l^* \} = 0.$$

Thus, by (4.24), as $\epsilon \to 0$,

$$
\begin{aligned}
&\sup\{||(-\triangle + 1 + V(\epsilon x))^{-1} f(u(\cdot - y)) \\
&\qquad -(-\triangle + 1)^{-1} f(u(\cdot - y))|| \mid u \in \overline{\mathcal{N}_{\delta_0,k}},\ y \in \overline{B_{\mathbb{R}^N}(0,R)},\ k \geq l^*\} \\
&\to 0.
\end{aligned}
$$

It follows that as $\epsilon \to 0$,

$$
\begin{aligned}
&\sup\{||\nabla E_\epsilon(u(\cdot - y)) - \nabla J(u(\cdot - y))|| \mid u \in \overline{\mathcal{N}_{\delta_0,k}},\ y \in \overline{B_{\mathbb{R}^N}(0,R)},\ k \geq l^*\} \\
&\to 0.
\end{aligned} \tag{4.25}
$$

Therefore, for $0 < r \leq r_0$, there exists $\epsilon_r > 0$ such that for any $u \in \overline{\mathcal{N}_{\delta_0,k}}$, $y \in \overline{B_{\mathbb{R}^N}(0,R)}$ and $k \geq l^*$,

$$
||\nabla E_\epsilon(u(\cdot - y)) - \nabla J(u(\cdot - y))|| < \frac{C}{4} r \ \text{ if } 0 \leq \epsilon \leq \epsilon_r, \tag{4.26}
$$

where the constant C comes from Lemma 4.1. Since $\nabla J(v(\cdot - y)) = \nabla J(v) = 0, \forall v \in \mathcal{K}$, we get that for any $0 < r \leq r_0$, there exists δ_r such that for any $0 < \delta \leq \delta_r$ and any $u \in N_{2\delta}(\mathcal{K})$,

$$
||\nabla J(u(\cdot - y))|| < \frac{C}{4} r. \tag{4.27}
$$

By (4.27) and the fact that (see (3.35))

$$
\lim_{k \to \infty} \overline{\mathcal{N}_{\delta,k}} \subset N_{2\delta}(\mathcal{K}),
$$

we deduce that there exists $k(\delta, r)$ such that if $k \geq k(\delta, r)$, then for any $0 < \delta \leq \delta_r$ and any $u \in \overline{\mathcal{N}_{\delta,k}}$,

$$
||\nabla J(u(\cdot - y))|| < \frac{C}{4} r. \tag{4.28}
$$

Thus, the claim follows from (4.26), (4.28) and the fact that

$$
||\Phi_{u,y,\epsilon,k}(0)|| \leq \frac{1}{C} ||\nabla E_\epsilon(u(\cdot - y))||.
$$

Combining (4.22) and (4.23) leads to

$$
||\Phi_{u,y,\epsilon,k}(w)|| \leq r
$$

for every $w \in \overline{B_{T^\perp_{u,y,k}}(0,r)}$. Therefore, $\Phi_{u,y,\epsilon,k}$ is a contractive mapping in $\overline{B_{T^\perp_{u,y,k}}(0,r)}$. By Banach fixed point theorem, there exists unique fixed point $w_{\delta,k}(u,y,\epsilon)$ of $\Phi_{u,y,\epsilon,k}$. Denote δ_{r_0} by δ^*, ϵ_{r_0} by ϵ^* and $k(\delta, r_0)$ by $k(\delta)$. It is easy to verify that the conclusions $(\mathbf{i}) - (\mathbf{iv})$ hold for $w_{\delta,k}(u,y,\epsilon)$.

Now, we prove that $w_{\delta,k} : \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)} \to Y$ is C^1. For any $(u_0, y_0) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}$ and (u,y) close to (u_0,y_0), both $S_{u_0,y_0,k}|_{T^\perp_{u,y,k}} : T^\perp_{u,y,k} \to T^\perp_{u_0,y_0,k}$ and $S_{u,y,k}|_{T^\perp_{u_0,y_0,k}} : T^\perp_{u_0,y_0,k} \to T^\perp_{u,y,k}$ are isomorphisms, and finding a solution $w \in T^\perp_{u,y,k}$ to the equation $S_{u,y,k} \nabla E_\epsilon(u(\cdot - y) + w) = 0$ is equivalent to finding a solution $w \in T^\perp_{u_0,y_0,k}$ to the equation $S_{u_0,y_0,k} S_{u,y,k} \nabla E_\epsilon(u(\cdot - y) + S_{u,y,k} w) = 0$. Note that $S_{u_0,y_0,k} S_{u,y,k} \nabla E_\epsilon(u(\cdot - y) + S_{u,y,k} w)$ is C^1 near $(u_0, y_0, w_0) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)} \times T^\perp_{u_0,y_0,k}$ and the Fréchet partial derivative of $S_{u_0,y_0,k} S_{u,y,k} \nabla E_\epsilon(u(\cdot - y) + S_{u,y,k} w)$ at (u_0, y_0, w_0) with respect to w is $L_{u_0,y_0,\epsilon,k}$ which is invertible. Therefore, the implicit functional theorem implies that

$$
w_{\delta,k}(\cdot,\cdot,\epsilon) : \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)} \to Y
$$

is C^1.

Finally, we give the proof of $(\mathbf{v})$. Let

$$
\varphi_{u,y,\epsilon,k} = u(\cdot - y) + w_{\delta,k}(u,y,\epsilon) - P_{T_{u,y,k}}(\nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))), \tag{4.29}
$$

where $P_{T_{u,y,k}}: Y \to T_{u,y,k}$ is orthogonal projection. By the conclusion (**ii**) of this Theorem, we get that

$$P_{T_{u,y,k}}(\nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))) = \nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)). \tag{4.30}$$

Thus, by (4.29) and (4.30), $\varphi_{u,y,\epsilon,k}$ satisfies

$$-\triangle\varphi_{u,y,\epsilon,k} + \varphi_{u,y,\epsilon,k} + V(\epsilon x)\varphi_{u,y,\epsilon,k} = f(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)). \tag{4.31}$$

By the definition of $T_{u,y,k}$, we have

$$\begin{aligned}
& P_{T_{u,y,k}}(\nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))) \\
= \; & \sum_{j=1}^N \Big\langle \nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)), \sum_{i=1}^s \xi_i(u)\frac{u_i(\cdot - y)}{\partial x_j}\Big\rangle \frac{\sum_{i=1}^s \xi_i(u)\frac{u_i(\cdot-y)}{\partial x_j}}{||\sum_{i=1}^s \xi_i(u)\frac{u_i(\cdot-y)}{\partial x_j}||^2} \\
& + \sum_{i=1}^k \langle \nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)), \tilde{e}_{i,k}(\cdot - y)\rangle \tilde{e}_{i,k}(\cdot - y) \\
& + \sum_{i=1}^q \langle \nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)), e_i(\cdot - y)\rangle e_i(\cdot - y).
\end{aligned} \tag{4.32}$$

Since $\tilde{e}_{i,k}$, e_i, u and $\frac{\partial u_i}{\partial x_j}$ satisfy exponential decay at infinity, by (4.32), for any given $k \geq k(\delta)$ and $n \geq 0$, there exists $C'_{n,k} > 0$ such that

$$\begin{aligned}
& \sup\{||(1+|x|)^n (P_{T_{u,y,k}}(\nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))))||_{L^\infty(\mathbb{R}^N)} \\
& \quad | \; u \in \overline{\mathcal{N}_{\delta,k}}, y \in \overline{B_{\mathbb{R}^N}(0,R)}, 0 \leq \epsilon \leq \epsilon^*\} \leq C'_{k,n}
\end{aligned} \tag{4.33}$$

and

$$\sup_{u \in \overline{\mathcal{N}_{\delta,k}}, y \in \overline{B_{\mathbb{R}^N}(0,R)}} ||(1+|x|)^n u(\cdot - y)||_{L^\infty(\mathbb{R}^N)} \leq C'_{k,n}. \tag{4.34}$$

Note that $\varphi_{u,y,\epsilon,k}$ satisfies the elliptic equation (4.31). Therefore, by the bootstrap argument and the fact that

$$\{w_{\delta,k}(u,y,\epsilon)) \mid u \in \overline{\mathcal{N}_{\delta,k}}, \; y \in \overline{B_{\mathbb{R}^N}(0,R)}, \; 0 \leq \epsilon \leq \epsilon^*\}$$

is compact in Y (because for fixed k, $\overline{\mathcal{N}_{\delta,k}}$ is compact), we get that

$$\sup\{||\varphi_{u,y,\epsilon,k}||_{L^\infty(\mathbb{R}^N)} \mid u \in \overline{\mathcal{N}_{\delta,k}}, \; y \in \overline{B_{\mathbb{R}^N}(0,R)}, \; 0 \leq \epsilon \leq \epsilon^*\} < \infty \tag{4.35}$$

and

$$\lim_{\rho\to\infty} \sup\{||\varphi_{u,y,\epsilon,k}||_{L^\infty(\mathbb{R}^N \setminus \overline{B_{\mathbb{R}^N}(0,\rho)})} \mid u \in \overline{\mathcal{N}_{\delta,k}}, \; y \in \overline{B_{\mathbb{R}^N}(0,R)}, \; 0 \leq \epsilon \leq \epsilon^*\} = 0. \tag{4.36}$$

By (4.35), (4.36) and (4.29), we get that

$$\sup\{||w_{\delta,k}(u,y,\epsilon)||_{L^\infty(\mathbb{R}^N)} \mid u \in \overline{\mathcal{N}_{\delta,k}}, \; y \in \overline{B_{\mathbb{R}^N}(0,R)}, \; 0 \leq \epsilon \leq \epsilon^*\} < \infty. \tag{4.37}$$

and

$$\begin{aligned}
& \lim_{\rho\to\infty} \sup\{||w_{\delta,k}(u,y,\epsilon)||_{L^\infty(\mathbb{R}^N \setminus \overline{B_{\mathbb{R}^N}(0,\rho)})} \mid u \in \overline{\mathcal{N}_{\delta,k}}, \; y \in \overline{B_{\mathbb{R}^N}(0,R)}, \; 0 \leq \epsilon \leq \epsilon^*\} \\
& = 0.
\end{aligned} \tag{4.38}$$

Let $d(t) = f(t)/t, t \in \mathbb{R}$. Then by (4.37), (4.34) and the condition $(\mathbf{F_1})$, we have

$$\begin{aligned}
& \sup\{||d(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))||_{L^\infty(\mathbb{R}^N)} \mid u \in \overline{\mathcal{N}_{\delta,k}}, \; y \in \overline{B_{\mathbb{R}^N}(0,R)}, \; 0 \leq \epsilon \leq \epsilon^*\} \\
& < \infty.
\end{aligned} \tag{4.39}$$

By the condition $(\mathbf{V_0})$, the condition $(\mathbf{F_1})$ and (4.38), we deduce that there exists ρ_0 such that

$$\inf\{1+V(\epsilon x)-d(u(x-y)+w_{\delta,k}(u,y,\epsilon)) \mid |x|>\rho_0,\ u\in\overline{\mathcal{N}_{\delta,k}},\\ y\in\overline{B_{\mathbb{R}^N}(0,R)},\ 0\le\epsilon\le\epsilon^*\}>0. \tag{4.40}$$

Let η be a cut-off function which satisfies that $\eta\equiv1$ in $B_{\mathbb{R}^N}(0,\rho_0)$ and $\eta\equiv0$ in $\mathbb{R}^N\setminus\overline{B_{\mathbb{R}^N}(0,\rho_0+1)}$. We can rewrite equation (4.31) as

$$\begin{aligned} & -\triangle\varphi_{u,y,\epsilon,k}+(1+V(\epsilon x)-(1-\eta(x))d(u(x-y)+w_{\delta,k}(u,y,\epsilon)))\varphi_{u,y,\epsilon,k} \\ = \ & f_{u,y,\epsilon,k}\end{aligned} \tag{4.41}$$

with

$$\begin{aligned} f_{u,y,\epsilon,k} \ = \ & d(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))\cdot u(\cdot-y) \\ & +\eta(x)\cdot d(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))\cdot w_{\delta,k}(u,y,\epsilon) \\ & -(1-\eta(x))\cdot d(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon)) \\ & \qquad \times(u(\cdot-y)-P_{T_{u,y,k}}(\nabla E_\epsilon(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))).\end{aligned} \tag{4.42}$$

By (4.34), (4.33), (4.39) and the fact that

$$\eta(x)d(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))\cdot w_{\delta,k}(u,y,\epsilon)$$

has compact support, we deduce that there exists $C'''_{n,k}>0$ such that

$$\sup_{u\in\overline{\mathcal{N}_{\delta,k}},y\in\overline{B_{\mathbb{R}^N}(0,R)}}||(1+|x|)^n f_{u,y,\epsilon,k}||_{L^\infty(\mathbb{R}^N)}\le C'''_{k,n}. \tag{4.43}$$

By (4.43), (4.40), (4.41) and [25, Proposition 4.2], we get that there exists $C''_{n,k}>0$ such that

$$\sup_{u\in\overline{\mathcal{N}_{\delta,k}},y\in\overline{B_{\mathbb{R}^N}(0,R)}}||(1+|x|)^n \varphi_{u,y,\epsilon,k}||_{L^\infty(\mathbb{R}^N)}\le C''_{k,n}. \tag{4.44}$$

Then the conclusion $(\mathbf{v})$ follows from (4.29), (4.44), (4.33) and (4.34). □

By the conclusion $(\mathbf{iii})$ of Theorem 4.2, we get that

$$J(u(\cdot-y)+w_{\delta,k}(u,y,0))\equiv I(u+\pi_k(u)),\ \forall(u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)}. \tag{4.45}$$

In what follows, for a C^1 mapping f defined in $\mathcal{N}_{\delta,k}\times B_{\mathbb{R}^N}(0,R)$, we use the the notations Df, D_uf and D_yf to denote the derivatives of f with respect to (u,y) variable, u variable and y variable respectively and use $Df(u,y)[\bar u,\bar y]$ to denote the derivative of f at the point (u,y) along the vector $(\bar u,\bar y)\in X_k\times\mathbb{R}^N$. Furthermore, we use $D_uf(u,y)[\bar u]$ and $D_yf(u,y)[\bar y]$ to denote the Fréchet partial derivatives with respect to the u and y variables along the vectors $\bar u$ and $\bar y$ respectively.

The condition $(\mathbf{V_1})$ for the potential V yields

$$\lim_{\epsilon\to0}\frac{V(\epsilon x)}{\epsilon^{n^*}}=Q_{n^*}(x). \tag{4.46}$$

The proof of the following proposition will be given in appendix.

Proposition 4.3. *Let $\delta>0$ be sufficiently small and $k\ge k(\delta)$. If $\iota<n^*$, then*

$$\lim_{\epsilon\to0}\sup\{\frac{1}{\epsilon^\iota}\Lambda_k(u,y,\epsilon)\mid(u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)}\}=0$$

where

$$\begin{aligned}\Lambda_k(u,y,\epsilon) \ = \ & ||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)|| \\ & +\sup_{\bar y\in\mathbb{R}^N,|\bar y|\le1}||Dw_{\delta,k}(u,y,\epsilon)[0,\bar y]-D(\pi_k(u)(\cdot-y))[0,\bar y]|| \\ & +\sup_{v\in X_k,||v||\le1}||Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]||.\end{aligned}$$

Moreover, there exists a constant $M > 0$ which is independent of (u, y) and ϵ such that for every $(u, y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0, R)}$ and $0 \le \epsilon \le \epsilon^*$,

$$\Lambda_k(u, y, \epsilon) \le M\epsilon^{n^*}.$$

For $0 < \delta \le \delta^*$ and $0 \le \epsilon \le \epsilon^*$, denote the functional

$$E_\epsilon(u(\cdot - y) + w_{\delta,k}(u, y, \epsilon)),\ (u, y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0, R)} \tag{4.47}$$

by $\Psi_k(u, y, \epsilon)$.

Theorem 4.4. *Suppose that $0 < \delta \le \delta^*$ and $k \ge k(\delta)$. Then there exists $\epsilon_k > 0$ such that if $0 \le \epsilon \le \epsilon_k$ and $(u_\epsilon, y_\epsilon) \in \mathcal{N}_{\delta,k} \times B_{\mathbb{R}^N}(0, R)$ is a critical point of the functional $\Psi_k(u, y, \epsilon)$, that is,*

$$D\Psi_k(u_\epsilon, y_\epsilon, \epsilon)[v, \bar{y}] = 0,\ \forall (v, \bar{y}) \in X_k \times \mathbb{R}^N, \tag{4.48}$$

then $u_\epsilon(\cdot - y_\epsilon) + w_{\delta,k}(u_\epsilon, y_\epsilon, \epsilon)$ is a critical point of E_ϵ.

Proof. By the conclusion ($\mathbf{ii}$) of Theorem 4.2 and hypothesis (4.48), we deduce that to prove $u_\epsilon(\cdot - y_\epsilon) + w_{\delta,k}(u_\epsilon, y_\epsilon, \epsilon)$ is a critical point of E_ϵ, it suffices to prove that for sufficiently small $\epsilon > 0$,

$$\begin{aligned} &\{v(\cdot - y_\epsilon) - (\bar{y} \cdot \nabla_x u_\epsilon)(\cdot - y_\epsilon) + Dw_{\delta,k}(u_\epsilon, y_\epsilon, \epsilon)[v, \bar{y}] \mid v \in X_k,\ \bar{y} \in \mathbb{R}^N\} \\ &+T^{\perp}_{u_\epsilon, y_\epsilon, k} = Y. \end{aligned} \tag{4.49}$$

If (4.49) were not true, then there exist $\epsilon_n \to 0$ as $n \to \infty$ such that $Y_n \neq Y$, where Y_n denotes the space appeared in the left side of (4.49) with $\epsilon = \epsilon_n$. Passing to a subsequence, we may assume that $y_{\epsilon_n} \to y_k$ and $u_{\epsilon_n} \to u_k$ in Y as $n \to \infty$, since $\{(u_{\epsilon_n}, y_{\epsilon_n})\}$ is a bounded sequence in the finite dimensional space $X_k \times \mathbb{R}^N$. By the hypothesis (4.48) and Proposition 4.3, we deduce that u_k is a critical point of $I(v + \pi_k(v))$. Then by the conclusion ($\mathbf{iv}$) of Lemma 3.8, $u_k + \pi_k(u_k)$ is a critical point of I. We denote it by $\tilde{u}_k$. Since $D\pi_k(u_k)v \in X$ and $\mathcal{T}_{u_k} \subset X^\perp$, we get $D\pi_k(u_k)v \perp \mathcal{T}_{u_k}$, where $\mathcal{T}_{u_k}$ comes from (3.24). Moreover, by Lemma 3.8, we get that $D\pi_k(u_k)v \in X_k^\perp$. Thus,

$$D\pi_k(u_k)v \perp X_k \oplus \mathcal{T}_{u_k} = T_{u_k,0,k}.$$

It follows that the following subspace of Y :

$$\{v - \bar{y}\nabla_x u_k - \bar{y}\nabla_x \pi_k(u_k) + D\pi_k(u_k)v \mid v \in X_k,\ \bar{y} \in \mathbb{R}^N\} + T^{\perp}_{u_k,0,k} \tag{4.50}$$

is equal to

$$\begin{aligned} &\{v - \bar{y}\nabla_x u_k - \bar{y}\nabla_x \pi_k(u_k) \mid v \in X_k,\ \bar{y} \in \mathbb{R}^N\} + T^{\perp}_{u_k,0,k} \\ = \ &\{v - \bar{y}\nabla_x \tilde{u}_k \mid v \in X_k,\ \bar{y} \in \mathbb{R}^N\} + T^{\perp}_{u_k,0,k}. \end{aligned} \tag{4.51}$$

As it has been mentioned above, $\tilde{u}_k = u_k + \pi_k(u_k) \in \mathcal{K}$. Therefore, by (3.3), we get that for every $1 \le j \le N$,

$$||\frac{\partial \tilde{u}_k}{\partial x_j} - \sum_{i=1}^{s} \xi_i(\tilde{u}_k)\frac{\partial u_i}{\partial x_j}|| \le \sum_{i=1}^{s} \xi_i(\tilde{u}_k)||\frac{\partial \tilde{u}_k}{\partial x_j} - \frac{\partial u_i}{\partial x_j}|| \le \varsigma. \tag{4.52}$$

By ($\mathbf{ii}$) of Lemma 3.8 and the fact that every ξ_i is a Lipschitz function, we deduce that for every $1 \le j \le N$, as $k \to \infty$,

$$\begin{aligned} &||\sum_{i=1}^{s} \xi_i(\tilde{u}_k)\frac{\partial u_i}{\partial x_j} - \sum_{i=1}^{s} \xi_i(u_k)\frac{\partial u_i}{\partial x_j}|| \\ \le \ &\sum_{i=1}^{s} |\xi_i(\tilde{u}_k) - \xi_i(u_k)| \cdot ||\frac{\partial u_i}{\partial x_j}|| \le C\sum_{i=1}^{s} ||\tilde{u}_k - u_k|| \cdot ||\frac{\partial u_i}{\partial x_j}|| \to 0, \end{aligned} \tag{4.53}$$

where C is the the Lipschitz constant of ξ_i. By (4.52) and (4.53), we obtain that for every $1 \leq j \leq N$,

$$\limsup_{k\to\infty} ||\frac{\partial \tilde{u}_k}{\partial x_j} - \sum_{i=1}^{s} \xi_i(u_k)\frac{\partial u_i}{\partial x_j}|| \leq \varsigma.$$

It follows that

$$\limsup_{k\to\infty} \sup_{|\bar{y}|\leq 1} ||\bar{y}\nabla_x \tilde{u}_k - \sum_{j=1}^{N} \bar{y}_j \sum_{i=1}^{s} \xi_i(u_k)\frac{\partial u_i}{\partial x_j}|| \leq \varsigma.$$

Thus, when ς is sufficiently small and k is sufficiently large, the space defined by (4.51) is equal to Y. As a consequence, when ς is sufficiently small and k is sufficiently large, the space defined by (4.50) is also Y. Therefore, the space

$$\begin{aligned}&\{v(\cdot - y_k) - (\bar{y}\nabla_x u_k)(\cdot - y_k) - (\bar{y}\nabla_x \pi_k(u_k))(\cdot - y_k) + (D\pi_k(u_k)v)(\cdot - y_k)\\ &\quad | \, v \in X_k, \ \bar{y} \in \mathbb{R}^N\} + T^{\perp}_{u_k,y,k}\end{aligned} \tag{4.54}$$

is equal to Y. Then we can define a bounded linear operator

$$\begin{aligned}&H_n : Y \to Y,\\ &w = v(\cdot - y_k) - (\bar{y}\nabla_x u_k)(\cdot - y_k) - (\bar{y}\nabla_x \pi_k(u_k))(\cdot - y_k) + (D\pi_k(u_k)v)(\cdot - y_k) + \phi\\ &\mapsto H_n(w) = v(\cdot - y_{\epsilon_n}) - (\bar{y}\nabla_x u_{\epsilon_n})(\cdot - y_{\epsilon_n}) + Dw_{\delta,k}(u_{\epsilon_n}, y_{\epsilon_n}, \epsilon_n)[v, \bar{y}] + \phi,\end{aligned}$$

where $\phi \in T^{\perp}_{u_k,y,k}$. It satisfies $Y_n = H_n(Y)$, where Y_n denotes the space appeared in the left side of (4.49) with $\epsilon = \epsilon_n$. By $u_{\epsilon_n} \to u_k$, $y_{\epsilon_n} \to y_k$ and Proposition 4.3, we get that as $n \to \infty$,

$$||H_n - id||_{\mathcal{L}(Y)} \to 0.$$

Therefore, when n is large enough, $H_n(Y) = Y$. It follows that $Y_n = Y$, which contradicts the assumption. Thus, when $k(\delta)$ is large enough and $k \geq k(\delta)$, there exists $\epsilon_k > 0$ such that if $0 \leq \epsilon \leq \epsilon_k$, then (4.49) holds. □

5 Proof of Theorem 1.3

By the conclusions $(\mathbf{iii})$ and $(\mathbf{v})$ of Theorem 4.2, if $u \in \overline{\mathcal{N}_{\delta,k}}$, then $\pi_k(u)$ decays exponentially at infinity. Therefore, for $u \in \overline{\mathcal{N}_{\delta,k}}$ and $y \in \mathbb{R}^N$, we can define

$$\Gamma_k(u, y) = \int_{\mathbb{R}^N} Q_{n^*}(x + y)(u + \pi_k(u))^2 dx.$$

By the same argument as Lemma 3.2 of [1] and by (4.46), (4.34) and the Lebesgue Convergence Theorem, we can get the following Lemma:

Lemma 5.1. *For any given $k \geq k(\delta)$, as $\epsilon \to 0$,*

$$\sup\Big\{\Big|\frac{1}{\epsilon^{n^*}}\int_{\mathbb{R}^N} V(\epsilon(x+y))(u+\pi_k(u))^2 dx - \Gamma_k(u,y)\Big| \,|\, (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}\Big\} \to 0$$

and

$$\begin{aligned}\sup\Big\{\Big|D\Big(\frac{1}{\epsilon^{n^*}}\int_{\mathbb{R}^N} V(\epsilon(x+y))(u+\pi_k(u))^2 dx - \Gamma_k(u,y)\Big)[v,\bar{y}]\Big| \,|\, v \in X_k, \ ||v|| \leq 1,\\ \bar{y} \in \mathbb{R}^N, \ |\bar{y}| \leq 1, \ (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}\Big\} \to 0.\end{aligned}$$

From now on, for the condition $(\mathbf{V_1})$, we always assume that $\triangle Q_{n^*} \geq 0$ and $\triangle Q_{n^*} \not\equiv 0$ in $\mathbb{R}^N$, since the proof for the other case is similar.

Lemma 5.2. *If $\delta > 0$ is small enough, then for any $u \in \overline{\mathcal{N}_{\delta,k}}$, $\Gamma_k(u,\cdot)$ has a strict local minimum at $y = 0$ and $D_y^2\Gamma_k(u,0)$ is a positive-definite matrix. More precisely, there exists a constant $A_k > 0$ such that*

$$D_y^2\Gamma_k(u,0)y \cdot y \geq A_k|y|^2, \ \forall u \in \overline{\mathcal{N}_{\delta,k}}, \ \forall y \in \mathbb{R}^N. \tag{5.1}$$

Proof. By Lemma 4.1 of [1], we know that $y = 0$ is a critical point of $\Gamma_k(u,\cdot)$ for every $u \in \overline{\mathcal{N}_{\delta,k}}$. If (5.1) were not true, then there exist $\delta_n > 0$, $u_n \subset \overline{\mathcal{N}_{\delta_n,k}}$, $n = 1, 2, \cdots$ and $\{y_n\} \subset S^{N-1}$ such that $\delta_n \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} |D_y^2\Gamma_k(u_n,0)y_n \cdot y_n| = 0. \tag{5.2}$$

Since (u_n, y_n) is bounded in the finite dimensional space $X_k \times \mathbb{R}^N$, passing to a subsequence, we may assume that $u_n \to u_0$ in X_k, and $y_n \to y_0 \in S^{N-1}$ as $n \to \infty$. Let $D_{ii}\Gamma_k(u_n, y)$ be the second derivative of $\Gamma_k(u_n, y)$ with respect to the variable y_i and diag$\{D_{11}\Gamma_k(u_n,0), \cdots, D_{NN}\Gamma_k(u_n,0)\}$ be diagonal matrix with diagonal elements $D_{11}\Gamma_k(u_n,0)$, $\cdots$, $D_{NN}\Gamma_k(u_n,0)$. By the appendix of [1], we get that

$$D_{ii}\Gamma_k(u_n,0) = -\frac{2}{N}\int_{\mathbb{R}^N}(u_n + \pi_k(u_n))\nabla Q_{n^*}(x)\cdot\nabla(u_n + \pi_k(u_n))dx, \ 1 \leq i \leq N. \tag{5.3}$$

Therefore,

$$\begin{aligned} D_y^2\Gamma_k(u_n,0)y_n \cdot y_n &= y_n^T \cdot \text{diag}\{D_{11}\Gamma_k(u_n,0),\cdots,D_{NN}\Gamma_k(u_n,0)\}\cdot y_n \\ &= -\frac{2}{N}|y_n|^2\int_{\mathbb{R}^N}(u_n+\pi_k(u_n))\nabla Q_{n^*}(x)\cdot\nabla(u_n+\pi_k(u_n))dx \\ &= -\frac{1}{N}|y_n|^2\int_{\mathbb{R}^N}\nabla Q_{n^*}(x)\cdot\nabla(u_n+\pi_k(u_n))^2dx \\ &= \frac{1}{N}|y_n|^2\int_{\mathbb{R}^N}\triangle Q_{n^*}(x)\cdot(u_n+\pi_k(u_n))^2dx \end{aligned} \tag{5.4}$$

By (5.2) and (5.4), we infer that

$$\lim_{n\to\infty} D_y^2\Gamma_k(u_n,0)y_n\cdot y_n = \frac{1}{N}|y_0|^2\int_{\mathbb{R}^N}\triangle Q_{n^*}(x)\cdot(u_0+\pi_k(u_0))^2dx = 0.$$

It is a contradiction, since we have assumed that $\triangle Q_{n^*}(x) \geq 0$ and $\triangle Q_{n^*} \not\equiv 0$ in $\mathbb{R}^N$. $\square$

In the rest of this section, we assume that $\delta > 0$ is sufficiently small and $k \geq k(\delta)$ is sufficiently large such that (3.57) holds, where the constant $k(\delta)$ comes from Theorem 4.2.

Proof of Theorem 1.3:

By definition of $\Psi_k(u,y,\epsilon)$ (see (4.47)), for $(u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}$,

$$\begin{aligned} & \Psi_k(u,y,\epsilon) \\ = \ & \frac{1}{2}||u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)||^2 + \frac{1}{2}\int_{\mathbb{R}^N}V(\epsilon x)|u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)|^2dx \\ & -\int_{\mathbb{R}^N}F(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))dx \\ = \ & \frac{1}{2}||u(\cdot - y) + w_{\delta,k}(u,y,0)||^2 + \frac{1}{2}||w_{\delta,k}(u,y,\epsilon) - w_{\delta,k}(u,y,0)||^2 \\ & +\langle u(\cdot - y) + w_{\delta,k}(u,y,0), w_{\delta,k}(u,y,\epsilon) - w_{\delta,k}(u,y,0)\rangle \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\mathbb{R}^N} V(\epsilon x)|u(\cdot - y) + w_{\delta,k}(u,y,0)|^2 dx \\
&+\frac{1}{2}\int_{\mathbb{R}^N} V(\epsilon x)|w_{\delta,k}(u,y,\epsilon) - w_{\delta,k}(u,y,0)|^2 dx \\
&+\int_{\mathbb{R}^N} V(\epsilon x)(u(\cdot - y) + w_{\delta,k}(u,y,0))\cdot(w_{\delta,k}(u,y,\epsilon) - w_{\delta,k}(u,y,0))dx \\
&-\int_{\mathbb{R}^N} F(u(\cdot - y) + w_{\delta,k}(u,y,0))dx \\
&-\int_{\mathbb{R}^N} f(u(\cdot - y) + w_{\delta,k}(u,y,0))\cdot(w_{\delta,k}(u,y,\epsilon) - w_{\delta,k}(u,y,0))dx \\
&-\eta_1(u,y,\epsilon), \qquad (5.5)
\end{aligned}
$$

where

$$
\begin{aligned}
&\eta_1(u,y,\epsilon) \\
= &\int_{\mathbb{R}^N} F(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))dx - \int_{\mathbb{R}^N} F(u(\cdot - y) + w_{\delta,k}(u,y,0))dx \\
&-\int_{\mathbb{R}^N} f(u(\cdot - y) + w_{\delta,k}(u,y,0))\cdot(w_{\delta,k}(u,y,\epsilon) - w_{\delta,k}(u,y,0))dx.
\end{aligned}
$$

By Taylor expansion, we deduce that there exists $0 < \theta = \theta(x) < 1$, $\forall x \in \mathbb{R}^N$ such that

$$
\begin{aligned}
\eta_1(u,y,\epsilon) &= \frac{1}{2}\int_{\mathbb{R}^N} f'(u(\cdot - y) + \theta w_{\delta,k}(u,y,0) + (1-\theta)w_{\delta,k}(u,y,\epsilon)) \\
&\qquad \times (w_{\delta,k}(u,y,\epsilon) - w_{\delta,k}(u,y,0))^2 dx
\end{aligned} \tag{5.6}
$$

By the condition $(\mathbf{F_1})$, Proposition 4.3 and (5.6), we deduce that

$$
\lim_{\epsilon\to 0}\sup\{\frac{1}{\epsilon^{n^*}}|\eta_1(u,y,\epsilon)| \mid (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}\} = 0. \tag{5.7}
$$

Note that for $v \in X_k$, $\bar{y} \in \mathbb{R}^N$,

$$
\begin{aligned}
&D\eta_1(u,y,\epsilon)[v,\bar{y}] \\
= &\int_{\mathbb{R}^N} f(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon)) \\
&\qquad \times (v(\cdot - y) - \bar{y}(\nabla_x u)(\cdot - y) + Dw_{\delta,k}(u,y,\epsilon)[v,\bar{y}])dx \\
&-\int_{\mathbb{R}^N} f(u(\cdot - y) + w_{\delta,k}(u,y,0)) \\
&\qquad \times (v(\cdot - y) - \bar{y}(\nabla_x u)(\cdot - y) + Dw_{\delta,k}(u,y,0)[v,\bar{y}])dx \\
&-\int_{\mathbb{R}^N} f'(u(\cdot - y) + w_{\delta,k}(u,y,0))\cdot(w_{\delta,k}(u,y,\epsilon) - w_{\delta,k}(u,y,0)) \\
&\qquad\quad \times (v(\cdot - y) - \bar{y}(\nabla_x u)(\cdot - y) + Dw_{\delta,k}(u,y,0)[v,\bar{y}])dx \\
&-\int_{\mathbb{R}^N} f(u(\cdot - y) + w_{\delta,k}(u,y,0))\cdot(Dw_{\delta,k}(u,y,\epsilon)[v,\bar{y}] - Dw_{\delta,k}(u,y,0)[v,\bar{y}])
\end{aligned} \tag{5.8}
$$

Then by the conclusion $(\mathbf{iii})$ of Theorem 4.2, Proposition 4.3 and the condition $(\mathbf{F_1})$, we deduce that

$$
\lim_{\epsilon\to 0}\sup\{\frac{1}{\epsilon^{n^*}}||D\eta_1(u,y,\epsilon)|| \mid (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}\} = 0. \tag{5.9}
$$

Combining (5.7) and (5.9) yields

$$
\lim_{\epsilon\to 0}\sup\{\frac{1}{\epsilon^{n^*}}(|\eta_1(u,y,\epsilon)| + ||D\eta_1(u,y,\epsilon)||) \mid (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}\} = 0. \tag{5.10}
$$

By the conclusion $(\mathbf{ii})$ of Theorem 4.2 and the fact that

$$w_{\delta,k}(u,y,\epsilon)-w_{\delta,k}(u,y,0)\in T^{\perp}_{u,y,k},$$

we get

$$\begin{aligned}&\langle u(\cdot-y)+w_{\delta,k}(u,y,0),w_{\delta,k}(u,y,\epsilon)-w_{\delta,k}(u,y,0)\rangle\\ =\ &\int_{\mathbb{R}^N} f(u(\cdot-y)+w_{\delta,k}(u,y,0))\cdot(w_{\delta,k}(u,y,\epsilon)-w_{\delta,k}(u,y,0))dx.\end{aligned}\tag{5.11}$$

By Proposition 4.3, we deduce that

$$\begin{aligned}&\eta_2(u,y,\epsilon)\\ :=\ &\frac{1}{2}||w_{\delta,k}(u,y,\epsilon)-w_{\delta,k}(u,y,0)||^2+\frac{1}{2}\int_{\mathbb{R}^N}V(\epsilon x)|w_{\delta,k}(u,y,\epsilon)-w_{\delta,k}(u,y,0)|^2dx\\ &+\int_{\mathbb{R}^N}V(\epsilon x)(u(\cdot-y)+w_{\delta,k}(u,y,0))(w_{\delta,k}(u,y,\epsilon)-w_{\delta,k}(u,y,0))dx\end{aligned}$$

also satisfies (5.10). By the conclusion $(\mathbf{iii})$ of Theorem 4.2, we infer that

$$J(u(\cdot-y)+w_{\delta,k}(u,y,0))=J(u(\cdot-y)+\pi_k(u)(\cdot-y))=I(u+\pi_k(u)).\tag{5.12}$$

Finally, by the conclusions $(\mathbf{iii})$ and $(\mathbf{v})$ of Theorem 4.2 and (4.34), we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N}V(\epsilon x)|u(\cdot-y)+w_{\delta,k}(u,y,0)|^2dx\\ =\ &\frac{1}{2}\int_{\mathbb{R}^N}V(\epsilon x)(u(\cdot-y)+\pi_k(u)(\cdot-y))^2dx\\ =\ &\frac{1}{2}\epsilon^{n^*}\Gamma_k(u,y)+\eta_3(u,y,\epsilon),\end{aligned}\tag{5.13}$$

where

$$\begin{aligned}\Gamma_k(u,y)&=\int_{\mathbb{R}^N}Q_{n^*}(x)(u(\cdot-y)+\pi_k(u)(\cdot-y))^2dx\\ &=\int_{\mathbb{R}^N}Q_{n^*}(x+y)(u+\pi_k(u))^2dx.\end{aligned}$$

By Lemma 5.1, the conclusion $(\mathbf{v})$ of Theorem 4.2 and (4.34), we deduce that η_3 satisfies (5.10). By $(5.5)-(5.13)$, we get that

$$\Psi_k(u,y,\epsilon)=I(u+\pi_k(u))+\frac{1}{2}\epsilon^{n^*}\Gamma_k(u,y)+\eta(u,y,\epsilon),\tag{5.14}$$

where $\eta=\eta_1+\eta_2+\eta_3$ satisfies (5.10).

By Lemma 5.2, for every $u\in\overline{\mathcal{N}_{\delta,k}}$, $\Gamma_k(u,y)$ has a strict local minimum at $y=0$ and there is a constant $A_k>0$ such that

$$D^2_y\Gamma_k(u,0)\geq A_k\mathrm{Id}\tag{5.15}$$

where Id denotes the $N\times N$ identity matrix. By (5.15) and (5.14), we deduce that there exists $\epsilon'_k>0$ such that if $0\leq\epsilon\leq\epsilon'_k$, then for every $u\in\overline{\mathcal{N}_{\delta,k}}$, there exists $y_\epsilon(u)\in B_{\mathbb{R}^N}(0,R/2)$ such that $y_\epsilon(u)$ is the unique minimizer of $\Psi_k(u,\cdot,\epsilon)$ in $B_{\mathbb{R}^N}(0,R)$. Moreover, by implicit functional theorem, $y_\epsilon(\cdot)\in C^1(\overline{\mathcal{N}_{\delta,k}})$. By (5.14), we get that

$$\lim_{\epsilon\to0}||\Psi_k(u,y_\epsilon(u),\epsilon)-I(u+\pi_k(u))||_{C^1(\overline{\mathcal{N}_{\delta,k}})}=0.\tag{5.16}$$

By [9, Theorem IV.3], a GM pair is a special kind of Conley index pair which is associated with some pseudo-gradient flow of a functional. Therefore, the GM pair (W_k,W_k^-) which was defined in Remark

3.11 is a Conley index pair associated with some pseudo-gradient flow of the functional $g_k(u) = I(u + \pi_k(u))$. Then by (5.16) and Theorem III.4 of [9], we deduce that if ϵ is small enough, then (W_k, W_k^-) is also a Conley index pair associated with some pseudo-gradient flow of the functional $\Psi_k(\cdot, y_\epsilon(\cdot), \epsilon)$. By (3.57) and Theorem 5.5.18 of [8], we infer that if ϵ is sufficiently small, then $\Psi_k(\cdot, y_\epsilon(\cdot), \epsilon)$ has at least a critical point $u_\epsilon \in \mathcal{N}_{\delta,k}$. Then by Theorem 4.4, $\tilde{u}_\epsilon := u_\epsilon(\cdot - y_\epsilon(u_\epsilon)) + w_{\delta,k}(u_\epsilon, y_\epsilon(u_\epsilon), \epsilon)$ is a critical point of E_ϵ. Moreover, by (5.16), we have

$$\lim_{\epsilon\to 0} \mathrm{dist}_Y(\tilde{u}_\epsilon, \mathcal{K}) = 0$$

with $\mathcal{K} = \mathcal{K}_a^b$. This finishes the proof of Theorem 1.3. □

6 Appendix A

In this appendix, we shall give the proof of the existence of $\{\tilde{e}_{j,k}\}$ which satisfies the conditions $(\mathbf{i})$ and $(\mathbf{ii})$ in Section 3.

Since $X \cap C_0^\infty(\mathbb{R}^N)$ is dense in X, for any $\mu_k > 0$, we can choose $\{\bar{e}_{j,k}\} \subset X \cap C_0^\infty(\mathbb{R}^N)$ such that

$$\sup_{1\le j\le k} ||\bar{e}_{j,k} - e'_j|| \le \mu_k \text{ and } ||\bar{e}_{j,k}|| = 1,\ 1 \le j \le k. \tag{6.1}$$

We show that if μ_k is small enough, then $\{\bar{e}_{j,k} \mid 1 \le j \le k\} \cup \{e_j \mid 1 \le j \le q\}$ is linearly independent. If it were not true, without loss of generality, we may assume that

$$\bar{e}_{k,k} = \sum_{j=1}^{k-1} \alpha_j \bar{e}_{j,k} + \sum_{j=1}^{q} \beta_j e_j, \tag{6.2}$$

then

$$\bar{e}_{k,k} = \sum_{j=1}^{k-1} \alpha_j e'_j + \sum_{j=1}^{k-1} \alpha_j(\bar{e}_{j,k} - e'_j) + \sum_{j=1}^{q} \beta_j e_j.$$

It follows that if $\mu_k < 1/4\sqrt{2}$, then

$$\begin{aligned}
1 = ||\bar{e}_{k,k}||^2 &= \sum_{j=1}^{k-1} \alpha_j^2 + ||\sum_{j=1}^{k-1} \alpha_j(\bar{e}_{j,k} - e'_j)||^2 + 2\langle \sum_{j=1}^{k-1} \alpha_j e'_j, \sum_{j=1}^{k-1} \alpha_j(\bar{e}_{j,k} - e'_j)\rangle \\
&\quad + \sum_{j=1}^{q} \beta_j^2 + 2\langle \sum_{j=1}^{q} \beta_j e_j, \sum_{j=1}^{k-1} \alpha_j(\bar{e}_{j,k} - e'_j)\rangle \\
&\ge \frac{3}{4}\sum_{j=1}^{k-1} \alpha_j^2 + \frac{3}{4}\sum_{j=1}^{q} \beta_j^2 + ||\sum_{j=1}^{k-1} \alpha_j(\bar{e}_{j,k} - e'_j)||^2 - 8\sum_{j=1}^{k-1} \alpha_j^2 ||\bar{e}_{j,k} - e'_j||^2 \\
&\ge \frac{1}{2}\sum_{j=1}^{k-1} \alpha_j^2 + \frac{1}{2}\sum_{j=1}^{q} \beta_j^2.
\end{aligned} \tag{6.3}$$

By (6.2),

$$e'_k = \sum_{j=1}^{k-1} \alpha_j e'_j + \sum_{j=1}^{k-1} \alpha_j(\bar{e}_{j,k} - e'_j) + \sum_{j=1}^{q} \beta_j e_j + (e'_k - \bar{e}_{k,k}),$$

combining (6.3), we get that

$$\begin{aligned}
1 = ||e'_k||^2 &= \sum_{j=1}^{k-1} \alpha_j \langle \bar{e}_{j,k} - e'_j, e'_k\rangle + \langle e'_k - \bar{e}_{k,k}, e'_k\rangle \le \mu_k \sum_{j=1}^{k-1} |\alpha_j| + \mu_k \\
&\le (\sqrt{2k} + 1)\mu_k.
\end{aligned}$$

This induces a contradiction if we assume $(\sqrt{2k}+1)\mu_k<1$. Thus, $\{\bar{e}_{j,k}\mid 1\le j\le k\}\cup\{e_j\mid 1\le j\le k\}$ is linearly independent if $\mu_k<\min\{1/(\sqrt{2k}+1),1/4\sqrt{2}\}$.

By (6.1) and

$$\langle\bar{e}_{j,k},\bar{e}_{j',k}\rangle=\langle e'_j+(\bar{e}_{j,k}-e'_j),e'_{j'}+(\bar{e}_{j',k}-e'_{j'})\rangle,\ \langle\bar{e}_{j,k},e_{j'}\rangle=\langle e'_j+(\bar{e}_{j,k}-e'_j),e_{j'}\rangle,$$

we get that

$$\sup_{1\le j,j'\le k,j\ne j'}|\langle\bar{e}_{j,k},\bar{e}_{j',k}\rangle|\le 2\mu_k+\mu_k^2,\ \sup_{j\ne j'}|\langle\bar{e}_{j,k},e_{j'}\rangle|\le\mu_k. \tag{6.4}$$

Therefore, if μ_k is sufficiently small, using Gram-Schmidt orthogonalizing process to $\{e_j\mid 1\le j\le q\}\cup\{\bar{e}_{j,k}\mid 1\le j\le k\}$, we get $\{\tilde{e}_{j,k}\mid 1\le j\le k\}$ which satisfies the conditions ($\mathbf{i}$) and ($\mathbf{ii}$) in Section 3.

7 Appendix B

In this appendix, we give the proof of Proposition 4.3.

Let

$$\eta_{u,y,k}=(-\triangle+1)^{-1}f(u(\cdot-y)+\pi_k(u)(\cdot-y)).$$

Then

$$\begin{aligned}\eta_{u,y,k}&=&(-\triangle+1+V(\epsilon x))^{-1}f(u(\cdot-y)+\pi_k(u)(\cdot-y)))\\&&+(-\triangle+1+V(\epsilon x))^{-1}V(\epsilon x)\eta_{u,y,k}.\end{aligned} \tag{7.1}$$

Subtracting equation

$$S_{u,y,k}\nabla E_\epsilon(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))=0$$

from equation

$$S_{u,y,k}\nabla J(u(\cdot-y)+\pi_k(u)(\cdot-y))=0,$$

by (7.1) and the mean value theorem, we get that

$$\begin{aligned}&&L_{u,y,\epsilon,k}\Big(w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)\Big)\\&=&-S_{u,y,k}(-\triangle+1+V(\epsilon x))^{-1}V(\epsilon x)\eta_{u,y,k}\\&&+S_{u,y,k}(-\triangle+1+V(\epsilon x))^{-1}\Big((f'(u(\cdot-y)+\tilde{w})-f'(u(\cdot-y)))\\&&\quad\times(w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y))\Big)\end{aligned} \tag{7.2}$$

where $\tilde{w}$ lies between $w_{\delta,k}(u,y,\epsilon)$ and $\pi_k(u)(\cdot-y)$. By the conclusion ($\mathbf{iv}$) of Theorem 4.2, we get that $||w_{\delta,k}(u,y,\epsilon)||\le r$ if $0<\delta\le\delta_r$ and $k\ge k(\delta)$. And by ($\mathbf{ii}$) of Lemma 3.8, we deduce that if $k(\delta)$ is large enough and $k\ge k(\delta)$, then $||\pi_k(u)(\cdot-y)||\le r$. Therefore, $||\tilde{w}||\le r$ if $0<\delta\le\delta_r$ and $k\ge k(\delta)$. Moreover, by (4.19), we deduce that if r is small enough, $0<\delta\le\delta_r$ and $k\ge k(\delta)$, then

$$\begin{aligned}&\Big|\Big|(-\triangle+1+V(\epsilon x))^{-1}\Big((f'(u(\cdot-y)+\tilde{w})-f'(u(\cdot-y))\cdot(w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y))\Big)\Big|\Big|\\&\le\frac{C}{2}||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||,\end{aligned} \tag{7.3}$$

where C is the constant appeared in Lemma 4.1. By (7.3), (7.2) and Lemma 4.1, we get that

$$C||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||\le 2||(-\triangle+1)^{-1}V(\epsilon x)\eta_{u,y,k}||. \tag{7.4}$$

By (4.34), the conclusion ($\mathbf{v}$) of Theorem 4.2 and [25, Proposition 4.2], we get that for any $n>0$,

$$\sup\{||(1+|x|)^n\eta_{u,y,k}||_{L^\infty(\mathbb{R}^N)}\mid(u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)}\}<\infty. \tag{7.5}$$

By (7.5), using the same argument as Lemma 3.2 of [1], we can get that if $\iota < n^*$,

$$\lim_{\epsilon\to 0}\{\int_{\mathbb{R}^N}\frac{V^2(\epsilon x)}{\epsilon^{2\iota}}\eta^2_{u,y,k}\mid (u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)}\}=0 \tag{7.6}$$

and

$$\sup\{\int_{\mathbb{R}^N}\frac{V^2(\epsilon x)}{\epsilon^{2n^*}}\eta^2_{u,y,k}\mid (u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)},\ 0\le\epsilon\le\epsilon^*\}<\infty.$$

Thus, for $\iota < n^*$,

$$\lim_{\epsilon\to 0}\sup\{\frac{1}{\epsilon^{\iota}}||(-\triangle+1)^{-1}V(\epsilon x)\eta_{u,y,k}||\mid (u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)}\}=0 \tag{7.7}$$

and

$$\sup\{\frac{1}{\epsilon^{n^*}}||(-\triangle+1)^{-1}V(\epsilon x)\eta_{u,y,k}||\mid (u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)},\ 0\le\epsilon\le\epsilon^*\}<\infty. \tag{7.8}$$

Combining (7.4), (7.7) and (7.8) yields that for $\iota < n^*$, if $\delta > 0$ is small enough and $k\ge k(\delta)$, then

$$\lim_{\epsilon\to 0}\{\frac{1}{\epsilon^{\iota}}||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||\mid (u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)}\}=0 \tag{7.9}$$

and

$$\begin{aligned}&\sup\{\frac{1}{\epsilon^{n^*}}||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||\mid (u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)},\ 0\le\epsilon\le\epsilon^*\}\\&<\infty.\end{aligned} \tag{7.10}$$

Recall that $S_{u,y,k}:Y\to T^{\perp}_{u,y,k}$ is orthogonal projection. Therefore, for $h\in Y$,

$$\begin{aligned}S_{u,y,k}h &= h-\sum_{j=1}^{q}\langle h,e_j(\cdot-y)\rangle e_j(\cdot-y)-\sum_{j=1}^{k}\langle h,\tilde e_{j,k}(\cdot-y)\rangle\tilde e_{j,k}(\cdot-y)\\ &\quad -\sum_{j=1}^{N}\Big\langle h,\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}(\cdot-y)\Big\rangle\frac{\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}(\cdot-y)}{||\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}||^2}.\end{aligned}$$

Thus, the Fréchet partial derivative of $S_{u,y,k}h$ with respect to u along the vector $v\in X_k$ is

$$\begin{aligned}&D_u(S_{u,y,k}h)[v]\\ =&\ -\sum_{j=1}^{N}\Big\langle h,\sum_{i=1}^{s}D\xi_i(u)[v]\cdot\frac{\partial u_i}{\partial x_j}(\cdot-y)\Big\rangle\frac{\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}(\cdot-y)}{||\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}||^2}\\ &-\sum_{j=1}^{N}\Big\langle h,\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}(\cdot-y)\Big\rangle\frac{\sum_{i=1}^{s}(D\xi_i(u)[v])\cdot\frac{\partial u_i}{\partial x_j}(\cdot-y)}{||\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}||^2}\\ &+2\sum_{j=1}^{N}\Big(\Big\langle h,\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}(\cdot-y)\Big\rangle\frac{\langle\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j},\sum_{i=1}^{s}(D\xi_i(u)[v])\frac{\partial u_i}{\partial x_j}\rangle}{||\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}||^4}\\ &\qquad\times\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}(\cdot-y)\Big)\end{aligned} \tag{7.11}$$

and the Fréchet partial derivative of $S_{u,y,k}h$ with respect to y along the vector $\bar y\in\mathbb{R}^N$ is

$$D_y(S_{u,y,k}h)[\bar y]$$

$$
\begin{aligned}
= \quad & \sum_{j=1}^{q}\langle h,(\bar{y}\nabla_x e_j)(\cdot-y)\rangle e_j(\cdot-y)+\sum_{j=1}^{k}\langle h,(\bar{y}\nabla_x\tilde{e}_{j,k})(\cdot-y)\rangle\tilde{e}_{j,k}(\cdot-y)\\
& +\sum_{j=1}^{q}\langle h,e_j(\cdot-y)\rangle(\bar{y}\nabla_x e_j)(\cdot-y)+\sum_{j=1}^{k}\langle h,\tilde{e}_{j,k}(\cdot-y)\rangle(\bar{y}\nabla_x\tilde{e}_{j,k})(\cdot-y)\\
& +\sum_{j=1}^{N}\Big\langle h,\sum_{i=1}^{s}\xi_i(u)\cdot(\bar{y}\nabla_x(\frac{\partial u_i}{\partial x_j}))(\cdot-y)\Big\rangle\frac{\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}(\cdot-y)}{||\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}||^2}\\
& +\sum_{j=1}^{N}\Big\langle h,\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}(\cdot-y)\Big\rangle\frac{\sum_{i=1}^{s}\xi_i(u)\cdot(\bar{y}\nabla_x(\frac{\partial u_i}{\partial x_j}))(\cdot-y)}{||\sum_{i=1}^{s}\xi_i(u)\frac{\partial u_i}{\partial x_j}||^2}. \qquad (7.12)
\end{aligned}
$$

Differentiating equations $S_{u,y,k}(\nabla E_\epsilon(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon)))=0$ and $S_{u,y,k}(\nabla J(u(\cdot-y)+\pi_k(u)(\cdot-y))=0$ with respect to the variable u along the vector $v\in X_k$, we get that

$$
\begin{aligned}
& S_{u,y,k}(\nabla^2 E_\epsilon(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))(v(\cdot-y)+Dw_{\delta,k}(u,y,\epsilon)[v,0]))\\
& +D_u(S_{u,y,k}h_1)[v]=0 \qquad (7.13)
\end{aligned}
$$

and

$$
\begin{aligned}
& S_{u,y,k}(\nabla^2 J(u(\cdot-y)+\pi_k(u)(\cdot-y))(v(\cdot-y)+D\pi_k(u)(\cdot-y)[v,0]))\\
& +D_u(S_{u,y,k}h_2)[v]=0, \qquad (7.14)
\end{aligned}
$$

where $h_1=\nabla E_\epsilon(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))$ and $h_2=\nabla J(u(\cdot-y)+\pi_k(u)(\cdot-y))$. By (7.1) and (7.3), it is easy to verify that there exists a constant $C>0$ such that

$$
||h_1-h_2||\le C||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||+C||(-\triangle+1)^{-1}V(\epsilon x)\eta_{u,y,k}||. \qquad (7.15)
$$

By (7.15) and (7.11), we get that for $||v||\le 1$, there exists a constant $C>0$ such that

$$
\begin{aligned}
& ||D_u(S_{u,y,k}h_2)[v]-D_u(S_{u,y,k}h_1)[v]||\\
\le \quad & C||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||+C||(-\triangle+1)^{-1}V(\epsilon x)\eta_{u,y,k}||. \qquad (7.16)
\end{aligned}
$$

A direct computation shows that

$$
\begin{aligned}
& S_{u,y,k}(\nabla^2 E_\epsilon(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))(v(\cdot-y)+Dw_{\delta,k}(u,y,\epsilon)[v,0]))\\
& -S_{u,y,k}(\nabla^2 J(u(\cdot-y)+\pi_k(u)(\cdot-y))(v(\cdot-y)+D\pi_k(u)(\cdot-y)[v,0]))\\
= \quad & S_{u,y,k}(\nabla^2 J(u(\cdot-y)+\pi_k(u)(\cdot-y))(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]))\\
& -S_{u,y,k}(-\triangle+1)^{-1}\Big\{\Big(f'(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))\\
& -f'(u(\cdot-y)+\pi_k(u)(\cdot-y))\Big)\times(v(\cdot-y)+Dw_{\delta,k}(u,y,\epsilon)[v,0])\Big\}\\
& +S_{u,y,k}(-\triangle+1)^{-1}V(\epsilon x)\bar{\eta}_{u,y,\epsilon,k}(v) \qquad (7.17)
\end{aligned}
$$

where

$$
\bar{\eta}_{u,y,\epsilon,k}(v)=(-\triangle+1+V(\epsilon x))^{-1}(f'(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon)))\cdot(v(\cdot-y)+Dw_{\delta,k}(u,y,\epsilon)[v,0])).
$$

By (4.34), the conclusion $(\mathbf{v})$ of Theorem 4.2 and (1.2) in $(\mathbf{F_1})$, we get that for any $v,h\in Y$, $||v||=||h||=1$,

$$
\begin{aligned}
& \int_{\mathbb{R}^N}\Big|f'(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))-f'(u(\cdot-y)+\pi_k(u)(\cdot-y))\Big|\\
& \qquad \times|v(\cdot-y)+Dw_{\delta,k}(u,y,\epsilon)[v,0]|\cdot|h|dx\\
\le \quad & C||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||.
\end{aligned}
$$

It follows that

$$\begin{aligned}&\Big|\Big|(-\triangle+1)^{-1}\Big\{\Big(f'(u(\cdot-y)+w_{\delta,k}(u,y,\epsilon))\\&\qquad -f'(u(\cdot-y)+\pi_k(u)(\cdot-y))\Big)\times(v(\cdot-y)+Dw_{\delta,k}(u,y,\epsilon)[v,0])\Big\}\Big|\Big|\\ \leq\;& C||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||.\end{aligned}\tag{7.18}$$

By (7.13), (7.14) and (7.16) $-$ (7.18), we deduce that

$$\begin{aligned}&||S_{u,y,k}(\nabla^2 J(u(\cdot-y)+\pi_k(u)(\cdot-y))(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]))||\\ \leq\;& C||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||+C||(-\triangle+1)^{-1}V(\epsilon x)\eta_{u,y,k}||\\&+C||(-\triangle+1)^{-1}V(\epsilon x)\bar\eta_{u,y,\epsilon,k}(v)||.\end{aligned}\tag{7.19}$$

By the conclusion (**ii**) of Lemma 3.8 and (4.19), we deduce that

$$\begin{aligned}\lim_{k\to\infty}\sup\Big\{&||\nabla^2 J(u(\cdot-y)+\pi_k(u)(\cdot-y))-\nabla^2 J(u(\cdot-y))||_{\mathcal{L}(Y)}\\&\mid (u,y)\in\overline{\mathcal{N}_{\delta,k}}\times\overline{B_{\mathbb{R}^N}(0,R)}\Big\}=0.\end{aligned}$$

Therefore, as $k\to\infty$,

$$\begin{aligned}&||S_{u,y,k}(\nabla^2 J(u(\cdot-y)+\pi_k(u)(\cdot-y))(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]))\\&-S_{u,y,k}(\nabla^2 J(u(\cdot-y))(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]))||\\&=o(1)||Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]||.\end{aligned}\tag{7.20}$$

By (7.19) and (7.20), we get that as $k\to\infty$,

$$\begin{aligned}&||S_{u,y,k}(\nabla^2 J(u(\cdot-y))(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]))||\\ \leq\;& C||w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y)||+C||(-\triangle+1)^{-1}V(\epsilon x)\eta_{u,y,k}||\\&+C||(-\triangle+1)^{-1}V(\epsilon x)\bar\eta_{u,y,\epsilon,k}(v)||\\&+o(1)||Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]||.\end{aligned}\tag{7.21}$$

Let $\mathcal{T}_u(\cdot-y)=\{h(\cdot-y)\mid h\in\mathcal{T}_u\}$ and $\mathcal{T}_u^{\perp}(\cdot-y)$ be the orthogonal complement space in Y, where $\mathcal{T}_u$ is defined in (3.24). Let $P_{\mathcal{T}_u^{\perp}(\cdot-y)}:Y\to\mathcal{T}_u^{\perp}(\cdot-y)$ and $P_{\mathcal{T}_u(\cdot-y)}:Y\to\mathcal{T}_u(\cdot-y)$ be orthogonal projections. Since $Dw_{\delta,k}(u,y,\epsilon)[v,0]\perp X_k(\cdot-y)$ and $D(\pi_k(u)(\cdot-y))[v,0]\perp X_k(\cdot-y)$, where $X_k(\cdot-y)=\{v(\cdot-y)\mid v\in X_k\}$, we deduce that

$$P_{\mathcal{T}_u^{\perp}(\cdot-y)}(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0])\in T^{\perp}_{u,y,k}.$$

Therefore, by Lemma 4.1, we have

$$\begin{aligned}&||S_{u,y,k}(\nabla^2 J(u(\cdot-y))P_{\mathcal{T}_u^{\perp}(\cdot-y)}(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]))||\\ =\;& ||L_{u,y,0,k}P_{\mathcal{T}_u^{\perp}(\cdot-y)}(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D\pi_k(u)(\cdot-y))[v,0])||\\ \geq\;& C||P_{\mathcal{T}_u^{\perp}(\cdot-y)}(Dw_{\delta,k}(u,y,\epsilon)[v,0]-D(\pi_k(u)(\cdot-y))[v,0]||.\end{aligned}\tag{7.22}$$

Differentiating the following equation with respect to variable u along the vector v,

$$\Big\langle w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y),\sum_{i=1}^{s}\xi_i(u)\frac{u_i(\cdot-y)}{\partial x_j}\Big\rangle=0$$

we get that

$$\begin{aligned}&\Big\langle D(w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y))[v,0],\sum_{i=1}^{s}\xi_i(u)\frac{u_i(\cdot-y)}{\partial x_j}\Big\rangle\\ =\;& -\Big\langle w_{\delta,k}(u,y,\epsilon)-\pi_k(u)(\cdot-y),\sum_{i=1}^{s}(D\xi_i(u)[v])\frac{u_i(\cdot-y)}{\partial x_j}\Big\rangle.\end{aligned}$$

It follows that there exists a constant $C > 0$ such that

$$\begin{aligned}&||P_{\mathcal{T}_u(\cdot - y)}(D(w_{\delta,k}(u,y,\epsilon) - \pi_k(u)(\cdot - y))[v,0])|| \\ \leq \quad & C||w_{\delta,k}(u,y,\epsilon) - \pi_k(u)(\cdot - y)||. \end{aligned} \tag{7.23}$$

By $(7.21) - (7.23)$, we deduce that when k is large enough, then there exists a constant $C > 0$ such that

$$\begin{aligned}&||D(w_{\delta,k}(u,y,\epsilon) - \pi_k(u)(\cdot - y))[v,0]|| \\ \leq \quad & C||w_{\delta,k}(u,y,\epsilon) - \pi_k(u)(\cdot - y)|| + C||(-\triangle + 1)^{-1}V(\epsilon x)\eta_{u,y,\epsilon,k}||. \\ & +C||(-\triangle + 1)^{-1}V(\epsilon x)\bar{\eta}_{u,y,\epsilon,k}(v)||. \end{aligned}$$

Then by $(7.7) - (7.10)$ and the fact that for $\iota < m$,

$$\begin{aligned}\lim_{\epsilon \to 0} \sup \Big\{ & \frac{1}{\epsilon^\iota}||(-\triangle + 1)^{-1}V(\epsilon x)\bar{\eta}_{u,y,\epsilon,k}(v)|| \\ & \quad | \ (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}, \ v \in X_k, \ ||v|| \leq 1 \Big\} = 0 \end{aligned}$$

and

$$\begin{aligned}\sup\{ & \frac{1}{\epsilon^{n^*}}||(-\triangle + 1)^{-1}V(\epsilon x)\bar{\eta}_{u,y,\epsilon,k}(v)|| \ | \ (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}, \\ & v \in X_k, \ ||v|| \leq 1, \ 0 \leq \epsilon \leq \epsilon^*\} < \infty, \end{aligned}$$

we get that for $\iota < n^*$,

$$\begin{aligned}\lim_{\epsilon \to 0} \sup \Big\{ & \frac{1}{\epsilon^\iota}||D(w_{\delta,k}(u,y,\epsilon) - \pi_k(u)(\cdot - y))[v,0]|| \\ & \quad | \ (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}, \ v \in X_k, \ ||v|| \leq 1 \Big\} = 0 \end{aligned} \tag{7.24}$$

and

$$\begin{aligned}\sup\{ & \frac{1}{\epsilon^{n^*}}||D(w_{\delta,k}(u,y,\epsilon) - \pi_k(u)(\cdot - y))[v,0]|| \ | \ (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}, \\ & v \in X_k, \ ||v|| \leq 1, \ 0 \leq \epsilon \leq \epsilon^*\} < \infty. \end{aligned} \tag{7.25}$$

Differentiating the two equations $S_{u,y,k}(\nabla E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))) = 0$ and $S_{u,y,k}(\nabla J(u(\cdot - y) + \pi_k(u)(\cdot - y)) = 0$ with respect to the variable y along the vector $\bar{y} \in \mathbb{R}^N$, we get that

$$\begin{aligned}& S_{u,y,k}(\nabla^2 E_\epsilon(u(\cdot - y) + w_{\delta,k}(u,y,\epsilon))(-\bar{y}\nabla_x u(\cdot - y) + Dw_{\delta,k}(u,y,\epsilon)[0,\bar{y}])) \\ & +D_y(S_{u,y,k}h_1)[\bar{y}] = 0 \end{aligned}$$

and

$$\begin{aligned}& S_{u,y,k}(\nabla^2 J(u(\cdot - y) + \pi_k(u)(\cdot - y))(-\bar{y}\nabla_x u(\cdot - y) + D(\pi_k(u)(\cdot - y))[0,\bar{y}])) \\ & +D_y(S_{u,y,k}h_2)[\bar{y}] = 0. \end{aligned}$$

The same arguments as (7.24) and (7.25) yield that for $\iota < n^*$,

$$\begin{aligned}\lim_{\epsilon \to 0} \sup \Big\{ & \frac{1}{\epsilon^\iota}||D(w_{\delta,k}(u,y,\epsilon) - \pi_k(u)(\cdot - y))[0,\bar{y}]|| \\ & \quad | \ (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}, \ \bar{y} \in \mathbb{R}^N, \ |\bar{y}| \leq 1 \Big\} = 0 \end{aligned}$$

and

$$\begin{aligned}\sup\{ & \frac{1}{\epsilon^{n^*}}||D(w_{\delta,k}(u,y,\epsilon) - \pi_k(u)(\cdot - y))[0,\bar{y}]|| \ | \ (u,y) \in \overline{\mathcal{N}_{\delta,k}} \times \overline{B_{\mathbb{R}^N}(0,R)}, \\ & \bar{y} \in \mathbb{R}^N, \ |\bar{y}| \leq 1, \ 0 \leq \epsilon \leq \epsilon^*\} < \infty. \end{aligned}$$

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